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BASIC MATHEMATICS FOR PHYSICS

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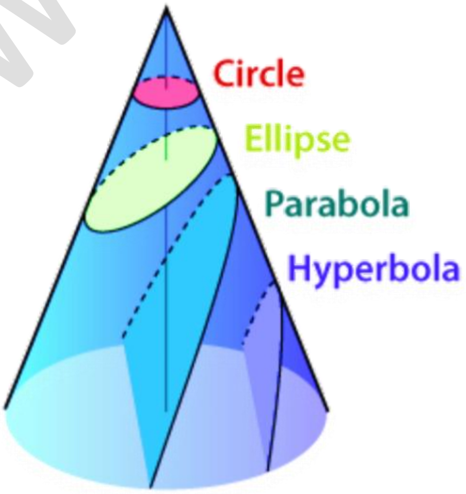
Topic 0 Basic Mathematics for Physics

CONTENTS

Topic
0.01 Logarithms 0.01.01 Indices 0.01.02 Logarithm 0.01.03 Laws of logarithms logarithm of 1 Change of base 0.01.04 Using log table Using log for calculation Antilog of negative number
0.02 Trigonometry 0.02.01 Definition of a radian 0.02.02 Trigonometric ratios for angles in a right- angled triangle 0.02.03 Sign of an angle in any quadrant 0.02.04 Trigonometric identities
0.03 Introduction to vectors 0.03.01 Scalar quantity 0.03.02 Vector quantities 0.03.03 Geometric Representation of vector 0.03.04 Position vector 0.03.05 Adding two vectors 0.03.06 Subtracting two vectors 0.03.07 Different types of vectors 0.03.08 Unit Vector or vector of unit length 0.03.09 Cartesian frame of reference 0.03.10 Vectors in two dimensions 0.03.11 Vectors in three dimensions 0.03.12 Polar representation of vector Cosine rule

	0.03.13 Displacement vector 0.03.14 Important result 0.03.15 Scalar product of vectors 0.03.16 Vector product or cross product
0.04 Differentiation	0.04.01 Constant Rate of Change 0.04.02 Rate of Change that is Not Constant 0.04.03 Derivatives of Polynomials 0.04.04 Linearity rules 0.04.05 Maxima and Minima of function 0.04.06 Distinguishing maximum points from minimum points
0.05 Integration	0.05.01 Definition 0.05.02 Application of integration for finding volume 0.05.03 Application of integration in Physics 0.05.04 Some important Integration formulas
0.06 Quadratic equation	



0.01 Logarithms

0.01.01 Indices

When a number is wrote in the form 2^4 , here 2 is known as base and 4 is known as power, index or exponent.

Rules of exponent

Consider we want to multiply 4 and 8 which is equal to 32

$4 \times 8 = 32$

Now $4 = 2^2$ and $8 = 2^3$.

As $4 \times 8 = 32$

$2^2 \times 2^3 = 32$

$(2\times2) \times (2\times2\times2) = 32$

$2^5 = 32$

From above we can conclude that if two number in exponential form, if their base is same then power or index or exponent gets added or

$a^m \times a^n = a^{(m+n)}$

Similarly it can be proved that

$a^m \div a^n = a^{(m-n)}$

Consider $(2^2)^3$

$(2^2)^3 = (2\times2)^3$

$(2^2)^3 = (2\times2) \times (2\times2) \times (2\times2)$

$(2^2)^3 = 2^6$

In general

$(a^m)^n = a^{(m\times n)}$

0.01.02 Logarithm

Consider the expression $16 = 2^4$. Remember that 2 is the base, and 4 is the power. An alternative, yet equivalent, way of writing this expression is $\log_2 16 = 4$.

This is stated as 'log of 16 to base 2 equals 4'.

We see that the logarithm is the same as the power or index in the original expression.

In general we can write

$x = a^m$ then $\log_a x = m$

From above

$10 = 10^1$ thus $\log_{10} 10 = 1$

Or $2 = 2^1$ thus $\log_2 2 = 1$

In general

$\log_a a = 1$

Exercises 0.1.01

1. Write the following using logarithms instead of powers

a) $8^2 = 64$ b) $3^5 = 243$ c) $2^{10} = 1024$ d) $5^3 = 125$

e) $10^6 = 1000000$ f) $10^{-3} = 0.001$ g) $3^{-2} = \frac{1}{9}$ h) $6^0 = 1$

i) $5^{-1} = \frac{1}{5}$ j) $\sqrt{49} = 7$ k) $27^{2/3} = 9$ l) $32^{-2/5} = 1/4$

2. Determine the value of the following logarithms

a) $\log_3 9$ b) $\log_2 32$ c) $\log_5 125$ d) $\log_{10} 10000$

e) $\log_4 64$ f) $\log_{25} 5$ g) $\log_8 2$ h) $\log_{81} 3$

i) $log_3\left(\frac{1}{27}\right)$ j) $\log_7 1$ k) $log_8\left(\frac{1}{8}\right)$ l) $\log_4 8$

m) $\log_a a^5$ n) $\log_c \sqrt{c}$ o) $\log_s s$ p) $log_e\left(\frac{1}{e^3}\right)$

0.01.03 Laws of logarithms

1) The first law of logarithms

Suppose

$x = a^n$ and $y = a^m$

then the equivalent logarithmic forms are

$\log_a x = n$ and $\log_a y = m$ …….(1)

Using the first rule of indices

$xy = a^{(n+m)}$

$\log_a xy = n+m$ and

from (1) and so putting these results together we have

$\log_a xy = \log_a x + \log_a y$

2) The second law of logarithms

Suppose $x = a^n$, or equivalently $\log_a x = n$. suppose we raise both sides of $x = a^n$ to the power m:

$X^m = (a^n)^m$

Using the rules of indices we can write this as

$x^m = a^{nm}$

Thinking of the quantity x^m as a single term, the logarithmic form is

$\log_a x^m = nm = m\log_a x$

This is the second law. It states that when finding the logarithm of a power of a number, this can be evaluated by multiplying the logarithm of the number by that power.

3) The third law of logarithms

As before, suppose

$x = a^n$ and $y = a^m$

with equivalent logarithmic forms

$\log_a x = n$ and $\log_a y = m$ (2)

Consider $x \div y$.

$$\frac{x}{y} = \frac{a^n}{a^m} = a^{(n-m)}$$

using the rules of indices.

In logarithmic form

$$log_a\left(\frac{x}{y}\right) = log_a a^{(n-m)}$$

$$log_a\left(\frac{x}{y}\right) = n - m$$

which from (2) can be written

$$log_a\left(\frac{x}{y}\right) = log_a x - log_a x$$

This is the third law.

The logarithm of 1

Recall that any number raised to the power zero is 1: $a^0 = 1$. The logarithmic form of this is $\log_a 1 = 0$

Change of base

Example

Suppose we wish to find $\log_5 25$.

This is the same as being asked 'what is 25 expressed as a power of 5 ?'

Now $5^2 = 25$ and so

$\log_5 25 = 2$.

Example

Suppose we wish to find $\log_{25} 5$.

This is the same as being asked 'what is 5 expressed as a power of 25 ?'

We know that 5 is a square root of 25, that is $5 = \sqrt{25}$. So $25^{1/2} = 5$

$$log_{25}5 = \frac{1}{2}$$

Notice from the last two examples that by interchanging the base and the number

$$log_5 25 = \frac{1}{log_{25}5}$$

In general

$$log_a b = \frac{1}{log_b a}$$

$$\frac{\log_e b}{\log_e a} = \frac{\log_c b}{\log_c a} = log_a b$$

Exercise 0.1.02

Each of the following expressions can be simplified to logN. Determine the value of N in each case. We have not explicitly written down the base. You can assume the base is 10, but the results are identical whichever base is used.

a) log 3 + log 5 b) log 16 − log 2 c) 3 log 4

d) 2 log 3 − 3 log 2 e) log 236 + log 1

f) log 236 − log 1 g) 5 log 2 + 2 log 5

h) log 128 − 7 log 2 i) log 2 + log 3 + log

j) log 12 − 2 log 2 + log 3

k) 5 log 2 + 4 log 3 − 3 log 4

l) log 10 + 2 log 3 − log 2

Common bases:

log means $\log_{10}$

ln means $\log_e$ where e is the exponential constant.

We can convert ln to log as follows

ln a = 2.303 log a

Exercises 0.1.03

Use logarithms to solve the following equations

a) $10^x = 5$ b) $e^x = 8$ c) $10^x = ½$ d) $e^x = 0.1$ e) $4^x = 12$ f) $3^x = 2$ g) $7^x = 1$

h) $\left(\frac{1}{2}\right)^x = \frac{1}{100}$

0.01.04 Using log table

Four figure logarithms

Logarithms can be used to calculate lengthy multiplication and division numerical

We can use log tables , for four figure logarithms.

Logarithm of number consists of two parts

Characteristic : Integral part of log

Mantissa : Fractional or decimal part of the log

Characteristic

If number is >1, then count number of digits before decimal, then reduce one from the number of digits

For example

6.234 : Number of digits before decimal is 1 ,

thus Characteristic number = 1-1 =0

62.34 : Number of digits before decimal are 2,

thus Characteristic number =2-1=1

623.4 : Number of digits before decimal are 3,

thus Characteristic number =3-1=2

6234.0 : Number of digits before decimal are 4,

thus Characteristic number =4-1=3

If number is <1, then count number of zero after decimal, then add one from the number of digits , and Characteristic number is negative represented as bar

For example

0.6234 : Number of zero's after decimal is zero ,

thus Characteristic number =-(0+1) = $\bar{1}$

0.0623: Number of zero's after decimal is 1 ,

thus Characteristic number = -(1+1) =$\bar{2}$

0.00623 : Number of zero's after decimal is 2 ,

thus Characteristic number = -(2+1) = $\bar{3}$

Exercises 0.1.04

Find characteristic number of following

a)523.045 b) 0.02569 c) 569325 d) 0.0023 e) 2.37×10^{3} f) 0.876

g) 2.569 h) 24.567 i) 0.00006 j) 1.236×10^{-3} k) 26.30×10^{-6} l) $.002\times10^{4}$

Ans

a) 2 b) $\bar{2}$ c) 5 d) $\bar{3}$ e) 3 f) $\bar{1}$ g) 0 h) 1 i) $\bar{5}$ j) $\bar{3}$ k) $\bar{5}$ l)1

Finding log of number using log table

Suppose we want log of number 1378 . characteristic number is 3

First Two digits

LOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	4	8	12	17	21	25	29	33	37
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	4	8	11	15	19	23	26	30	34
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	3	7	10	14	17	21	24	28	31
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	3	6	10	13	16	19	23	26	29
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	3	6	9	12	15	18	21	24	27
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25

Third digit

fourth digit

As shown in figure first two digits from left column, third digit in middle and fourth digit in right column

Now from table log of 137 refers to 1367 now add mean difference 26

We get 1367 +26 = 1393

Thus log 1378 = 3.1393

Log of 137.8 = 2.1393 (note only characteristic number changed to 2)

Log 13.78 = 1.1393 (note only characteristic number changed to 1)

Log 1.378 = 0.1393 (note only characteristic number changed to 0)

Log 0.1378 = $\bar{1}$.1393 (note only characteristic number changed to -1)

Log 0.01378 = $\bar{2}$.1393 (Note that zeros after decimal is omitted while finding log and characteristic number changed)

log5 = 0.6990 [note in table look for 50 = 6990, but characteristic is 0]

log50 = 1.6990

Finding Antilog of number

First Two digits

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference 1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
.01	1023	1026	1028	1030	1033	1035	1038	1040	1042	1045	0	0	1	1	1	1	2	2	2
.02	1047	1050	1052	1054	1057	1059	1062	1064	1067	1069	0	0	1	1	1	1	2	2	2
.03	1072	1074	1076	1079	1081	1084	1086	1089	1091	1094	0	0	1	1	1	1	2	2	2
.04	1096	1099	1102	1104	1107	1109	1112	1114	1117	1119	0	1	1	1	1	2	2	2	2
.05	1122	1125	1127	1130	1132	1135	1138	1140	1143	1146	0	1	1	1	1	2	2	2	2
.06	1148	1151	1153	1156	1159	$\boxed{1161}$	1164	1167	1169	1172	0	1	1	1	1	2	2	$\boxed{2}$	2
.07	1175	1178	1180	1183	1186	1189	1191	1194	1197	1199	0	1	1	1	1	2	2	2	2
.08	1202	1205	1208	1211	1213	1216	1219	1222	1225	1227	0	1	1	1	1	2	2	2	3
.09	1230	1233	1236	1239	1242	1245	1247	1250	1253	1256	0	1	1	1	1	2	2	2	3
.10	1259	1262	1265	1268	1271	1274	1276	1279	1282	1285	0	1	1	1	1	2	2	2	3

Third digit

fourth digit

For Antilog of any number, digits after decimal is considered

For example

Antilog (3.0658)

Here 3 is characteristic number hence should not be considered. Antilog of 0.0658 will be 1161 +2 = 1163 as shown in figure

Now put decimal point after one digit from left = 1.163

Characteristic number 3 will be now power of 10 thus final antilog will be

Antilog (3.0658) =1.163×10^3

Antilog($\bar{1}.0658$)

As stated earlier Antilog of 0.0658 will be 1161 +2 = 1163 as shown in figure

Now put decimal point after one digit from left = 1.163

Characteristic number $\bar{1}$ will be now power of 10 thus final antilog will be

Antilog ($\bar{1}$.0658) =1.163×10^{-1}

Similarly antilog of $\bar{5}.0658$ = 1.163×10^{-5}

To find antilog of 6.4632

ANTILOGARITHMS

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
.00	1000	1002	1005	1007	1009	1012	1014	1016	1019	1021	0	0	1	1	1	1	2	2	2
– –	– –	– –	– –	– –	– –	– –	– –	– –	– –	– –	–	–	–	–	–	–	–	–	–
.41	2570	2576	2582	2588	2594	2600	2606	2612	2618	2624	1	1	2	2	3	4	4	5	5
.42	2630	2636	2642	2649	2655	2661	2667	2673	2679	2685	1	1	2	2	3	4	4	5	6
.43	2692	2698	2704	2710	2716	2723	2729	2735	2742	2748	1	1	2	3	3	4	4	5	6
.44	2754	2761	2767	2773	2780	2786	2793	2799	2805	2812	1	1	2	3	3	4	4	5	6
.45	2818	2825	2831	2838	2844	2851	2858	2864	2871	2877	1	1	2	3	3	4	5	5	6
.46	2884	2891	2897	$\boxed{2904}$	2911	2917	2924	2931	2938	2944	1	$\boxed{1}$	2	3	3	4	5	5	6
.47	2951	2958	2965	2972	2979	2985	2992	2999	3006	3013	1	1	2	3	3	4	5	5	6
.48	3020	3027	3034	3041	3048	3055	3062	3069	3076	3083	1	1	2	3	4	4	5	6	6
.49	3090	3097	3105	3112	3119	3126	3133	3141	3148	3155	1	1	2	3	4	4	5	6	6
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

As stated earlier 6 is characteristic number , which is power of 10 and will be dropped

From table above antilog of 4632 = 2904+1 =2905 =2.905

Thus antilog(6.4632)=2.905 ×10^{6}

Similarly antilog ($\bar{3}.4632$) = 2.905 ×10^{-3}

Using log for calculation

To calculate value of y

$$y = \frac{4568 \times 3258}{0.02568}$$

Take log on both sides

$$\log y = \log\left(\frac{4568 \times 3258}{0.02568}\right)$$

Using log rules we get

Log y = log 4568 + log 3258 – log 0.02568 ----(1)

Now log 4568 , characteristic number 3

From log table 6590+8 =6598

Thus log(4568) = 3.6598

Similarly log(3258) = 3.5130

From equation (1) we have to add log of 4568 and 3258 thus

$$\begin{array}{rl} & \log(4568) = 3.6598 \\ + & \log(3258) = 3.5130 \\ \hline & 7.1728 \end{array}$$

Now log (0.025686) = $\bar{2}.4097$ (note we have round-off number as 0.02569)

From equation (i) we have to subtract log from previous log value 7.1728

$$\begin{array}{r} 7.1728 \\ - \ \bar{2}.4097 \\ \hline 8.7631 \end{array}$$

Note subtraction of characteristic number 7-1 =6 because of carry forward. Then 6 –(2) = 8.

Now we will find antilog of 8.7631. Digit 8 which is before decimal point refers to characteristic number and will be power of ten. From antilog table we get,

Antilog (8.7631) = 5.795×10^8 which the value of y.

∴ y = 5.795×10^8

Example

$$y = 125^{\frac{1}{6}}$$

Take log on both sides,

$$\log y = \log(125)^{\frac{1}{6}}$$

By applying log rule,

$$\log y = \frac{1}{6}\log(125)$$

From log table log125 = 2.0969

Now divide 2.0969 by 6 we get 0.3495 (after round-off)

Now take antilog of 0.3495 from antilog log table we get

Antilog(0.3495) =2.237×10^0

Thus $125^{\frac{1}{6}} = 2.237$

Exercises 0.1.05

Solve

a) $39^{3/4}$ b) $25^{1/3}$ c) $5^{1/2}$

d) $\frac{0.369\times0.0569}{0.00235}$ e) $\frac{(2.569\times10^7)\times(3.421\times10^{-4})}{45689}$

Answer

a) 15.61 b) 2.924 c)2.237 d) 8.933 e) 0.1923

Antilog of negative number

such as -7.5231

First convert the negative number in to two parts one negative (Characteristic) and the decimal part(Mantissa) into positive.

by adding +8 and subtracting -8 as follows :

-7.5231 +8 - 8 = -8 + (8-7.5231) = -8+ 0.4769.

Find the actual digits using 0.4769 in the anti-log table. Multiply this by 10^{-8} to account for the -8 characteristic

Ans : 2.998×10^{-8}

Example 2

-12.7777

-12.7777 +13-13 = -13 +(13-12.7777) = -13 +0.2223

Find the actual digits using 0.2223 in the anti-log table. Multiply this by 10^{-13} to account for the -13 characteristic

Ans 1.668×10^{-13}

Exercises 0.1.06

Find antilog of following

a) -2.5689 b) -6. 9945 c) -3. 1129

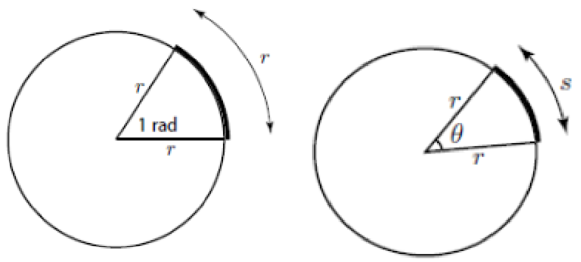
Answers

a) 2.699×10^{-3} b) 1.013×10^{-7} c) 7.711×10^{-4}

0.02 Trigonometry

0.02.01 Definition of a radian

Consider a circle of radius *r* as shown, In Figure we have highlighted part of the circumference of the circle chosen to have the same length as the radius. The angle at the centre, so formed, is 1 radian. Length of arc s = rθ Here θ is in radians



Exercises 0.2.01

Determine the angle (in radians) subtended at the centre of a circle of radius 3cm by each of the following arcs:

a) Arc of length 6 cm b) arc of length 3π cm

c) Arc of length 1.5 cm d) arc of length 6π cm

Answers

a)2 b) π c) 0.5 d) 2π

Equivalent angles in degrees and in radians

We know that the arc length for a full circle is the same as its circumference, $2\pi r$.

We also know that the arc length = $r\theta$.

So for a full circle

$2\pi r = r\theta$

$\theta = 2\pi$

In other words, when we are working in radians, the angle in a full circle is 2π radians, in other words

$360^{0} = 2$ *radians*

This enables us to have a set of equivalences between degrees and radians

Exercises 0.2.02

1. Convert angle in radians

a)90^{O} b) 360^{O} c) 60^{O} d) 45^{O} e) 120^{O} f) 15^{o} g) 30^{o} h) 270^{o}

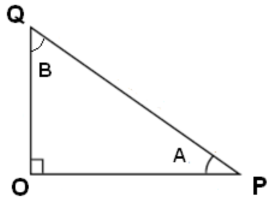
2. Convert radians to degrees

a) π/2 radians b) 3 π /4 radians c) π radians d) π /6 radians

e) 5 π radians f) 4 π /5 radians g) 7 π /4 radians h) π /10 radians

0.02.02 Trigonometric ratios for angles in a right-angled triangle

Refer to the triangle in Figure 1.



The side opposite the right-angle is called the hypotenuse

Recall the following important definitions:

$sinA = \frac{OQ}{QP}$ but $cosB = \frac{OQ}{QP}$

$\therefore$ sin A = cos B

We know that ∠ A + ∠ B = 90 $\therefore$ ∠ B = 90 - ∠ A

$\therefore$ sin A = cos (90-A)

$cosA = \frac{OP}{QP}$ but $sinB = \frac{OP}{QP}$

$\therefore$ cos A = sin B

OR cos A = sin (90-A)

$tanA = \frac{OQ}{OP}$ but $cotB = \frac{OQ}{OP}$

$\therefore$ tan A = cot B OR tan A = cot (90-A)

Angles

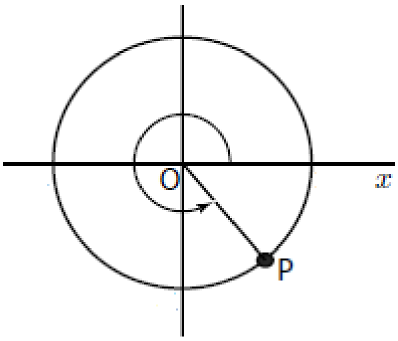


figure (a)

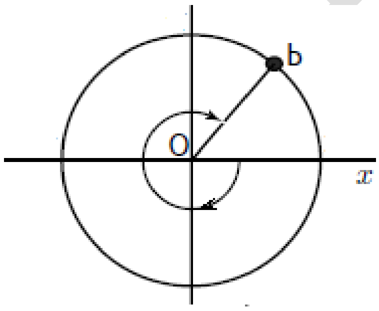


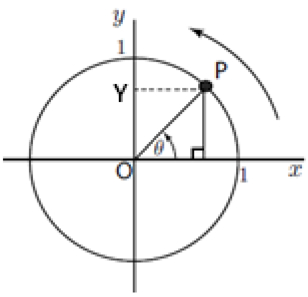
figure (b)

If angle is measured in anticlockwise direction from positive x-axis as shown in figure a. is positive If angle is measured in clockwise direction from positive x-axis as shown in figure b is negative

0.02.03 The sign of an angle in any quadrant

Sin of an angle in first quadrant

Consider Figure which shows a circle of radius 1 unit.



The side opposite θ has the same length as the projection of OP onto the y axis OY.

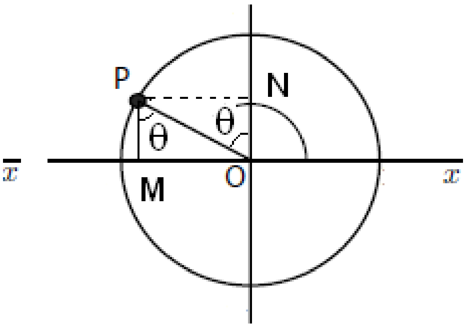
The arm OP is in the first quadrant and we have dropped a perpendicular line drawn from P to the x axis in order to form the right-angled triangle shown.

Consider angleθ. The side opposite this angle has the same length as the projection of OP onto the y axis. So we define

$$sin\theta = \frac{Projection\ of\ OP\ on\ y - axis}{OP}$$

Sinθ = Projection of OP on y axis

Sin of an angle in second quadrant



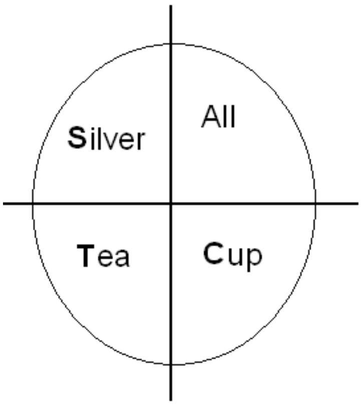
Consider adjacent figure here OP makes angle is 90 + θ with positive x-axis

Now as stated earlier sin(90+θ) = Projection of OP on y-axis = ON

From the geometry of figure we can find that $cos\theta = \frac{PM}{OP}$

Thus sin(90+θ) = cos θ

By using above we can obtain various relations, which can be quickly remembered by following way



Quadrant I: All

All ratios sin, cos, tan, cosec, sec, cot have POSITIVE value

Quadrant II: Silver

Only sine or cosec have POSITIVE value

Remaining have negative value

Quadrant III: Tea

Only tan and cot have POSITIVE value .Remaining have negative value

Quadrant IV: Cup

Only cos and sec have POSITIVE value

Remaining have negative values

Angles π + θ function do not change

For example sin(π +θ) = -sinθ

Here π +θ is in Third quadrant where sin is NEGATIVE thus negative sign appears

For angle $\left(\frac{\pi}{2}+\theta\right)$ and $\left(\frac{3\pi}{2}+\theta\right)$function changes from

sin → cos ; sec → cosec ; cosec → sec;

tan → cot; cos →sin ; cot → tan

And the sign of resultant depends on the quadrant of the first function

Sin(-θ)	Angle in IV quadrant Sin is negative	-sinθ
Cos (-θ)	Angle in IV quadrant cos is positive	cosθ
tan (-θ)	Angle in IV quadrant tan is negative	-tanθ
Cot (-θ)	Angle in IV quadrant cot is negative	-cotθ
$sin\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant sin positive and change function	cosθ
$cos\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant cos negative and change function	-sinθ
$tan\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant tan negative and change function	-cotθ
$cot\left(\frac{\pi}{2}+\theta\right)$	Angle is in II quadrant cot negative and change function	-tanθ
sin(π+θ)	Angle is in III quadrant sin negative and same function	-sinθ
cos (π+θ)	Angle is in III quadrant cos negative and same function	-cosθ
tan(π+θ)	Angle is in III quadrant tan positive and same function	tan θ
cot(π+θ)	Angle is in III quadrant cot positive and same function	cotθ

By using same technique you may find formula for

(π − θ) ,

$\left(\frac{3\pi}{2}+\theta\right)$ and

$\left(\frac{3\pi}{2}-\theta\right)$

0.02.04 Trigonometric identities

$$Sin^2\theta + cos^2\theta = 1$$

$$1 + tan^2\theta = sec^2\theta$$

$$1 + cot^2\theta = cosec^2\theta$$

$$sec^2\theta - tan^2\theta = 1$$

$$cosec^2\theta - cot^2\theta = 1$$

$$sin2\theta = 2sin\theta\, cos\theta = \frac{2\tan\theta}{1 + tan^2\theta}$$

$$cos2\theta = cos^2\theta - sin^2\theta = 2\cos^2\theta - 1 = 1 - 2sin^2\theta = \frac{1 - tan^2\theta}{1 + tan^2\theta}$$

$$sin(\alpha \pm \beta) = sin\alpha \cos\beta \pm \cos\alpha\, sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp sin\alpha\, sin\beta \text{ (note sign changed)}$$

$$sin\alpha + sin\beta = 2sin\left(\frac{\alpha+\beta}{2}\right) cos\left(\frac{\alpha-\beta}{2}\right)$$

$$sin\alpha - sin\beta = 2cos\left(\frac{\alpha+\beta}{2}\right) sin\left(\frac{\alpha-\beta}{2}\right)$$

$$cos\alpha + cos\beta = 2\, cos\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$cos\alpha - cos\beta = -2\, sin\left(\frac{\alpha+\beta}{2}\right) \sin\left(\frac{\alpha-\beta}{2}\right)$$

$$2sin\,\alpha\cos\beta = sin(\alpha+\beta) + sin(\alpha-\beta)$$

$$2\cos\alpha\, sin\,\beta = sin(\alpha+\beta) - sin(\alpha-\beta)$$

$$2\cos\alpha\cos\beta = \cos(\alpha+\beta) + \cos(\alpha-\beta)$$

$$-2\sin\alpha\sin\beta = \cos(\alpha+\beta) - \cos(\alpha-\beta)$$

$$Sin\,3\theta = 3\sin\theta - 4\sin^3\theta$$

$$Cos\,3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - tan^2\theta}$$

0.03 Introduction to vectors

0.03.01 Scalar quantity

Scalars are quantities that are fully described by a magnitude (or numerical value) alone.

For example if I get 5 numbers of apple from east direction and 3 numbers of apple from west direction number of apple I will have is 5+3 = 8 numbers of apples. Thus total number of apples does not depends on the directions.

Now if I say I want 5 water. It does not make any sense, as I have not mentioned any unit. Is it 5 litre or 5 ml or 5 kL of water

Thus scalar quantities have unit and magnitude (number) to give full description and is independent of direction

Some examples of scalars include the mass, charge, volume, time, speed, temperature, electric potential at a point inside a medium, energy .

The distance between two points in three-dimensional space is a scalar, but the direction from one of those points to the other is not, since describing a direction requires two physical quantities such as the angle on the horizontal plane and the angle away from that plane. Force cannot be described using a scalar, since force is composed of direction and magnitude, however, the magnitude of a force alone can be described with a scalar, for instance the gravitational force acting on a particle is not a scalar, but its magnitude is. The speed of an object is a scalar (e.g. 180 km/h), while its velocity is not (i.e. 180 km/h *north*). Other examples of scalar quantities in Newtonian mechanics include electric charge and charge density.

0.03.02 Vector quantities

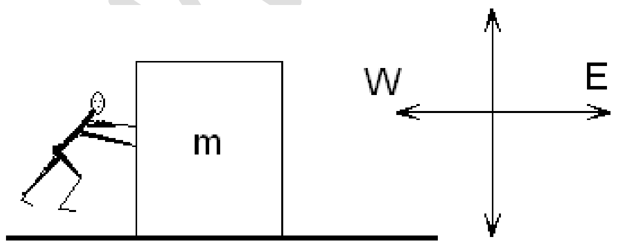


figure a

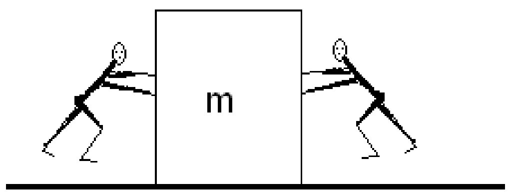


figure b

Vectors are quantities that are fully described by both a magnitude and a direction.

As shown in figure a, person is pushing a box of mass m towards East. Box will move in direction towards East. If F is the force applied by a man then acceleration of box is F/m and towards East

Now as shown in figure b, first person is pushing a box towards East while another person pushing box towards West. Now if both persons apply equal force then box does not move. As force towards East is cancelled or nullified by the force towards West.

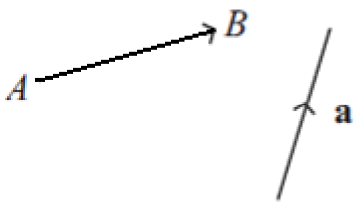
Thus to know the effective force on the box we should know the direction of force. If in above example if both person push the box in same direction(say towards East) then their forces would have got added and box would have started to move towards East.

Thus to understand vector quantity and its effect on object or at a particular point, we not only require magnitude but also direction such quantity are call vector quantity.

Some examples of vectors include weight as it is a gravitational force on object, velocity, acceleration, Electric filed, Magnetic field, Area,

0.03.03 Geometric Representation of vector

We can represent a vector by a line segment. This diagram shows two vectors.

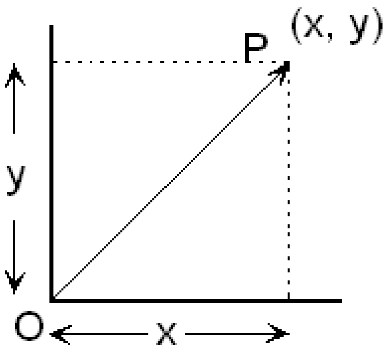


We have used a small arrow to indicate that the first vector is pointing from A to B. A vector pointing from B to A would be going in the opposite direction. Sometimes we represent a vector with a small letter such as a, in a bold typeface. This is common in textbooks, but it is inconvenient in handwriting. In writing, we normally put a bar underneath, or sometimes on top of, the letter: $\bar{a}$ or $\vec{a}$. In speech, we call this the vector "a-bar".

Arrow at B shows the direction called as head of vector, Length of arrow length AB is magnitude of vector let it be a , while point A is called as tail of vector.

Vector may be represented as $\overrightarrow{AB}$ it indicates tail of vector is at A and head is at B. If two vector p and q are equal of $\vec{p} = \vec{q}$ it means both the vectors have same magnitude and same direction.

0.03.04 Position vector

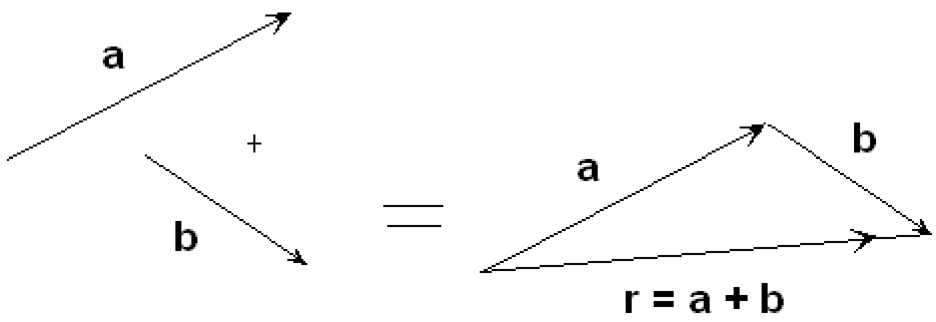


Vector OP or $\overrightarrow{OP}$ is called a position vector as it shows position of point P in co-ordinate frame. Its tail is at origin. It may be also represented as (x, y)

0.03.05 Adding two vectors

One of the things we can do with vectors is to add them together. We shall start by adding two vectors together. Once we have done that, we can add any number of vectors together by adding the first two, then adding the result to the third, and so on.

In order to add two vectors, we think of them as displacements. We carry out the first displacement, and then the second. So the second displacement must start where the first one finishes.

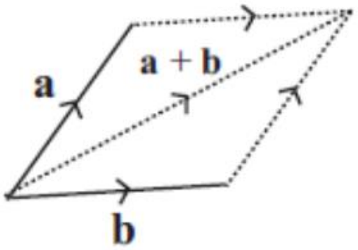


To add vector b in vector a, we have drown vector b from the head of a, keeping direction and magnitude same as of b. Then drawn vector from tail of vector a to head of vector b

The sum of the vectors, a + b (or the *resultant*, as it is sometimes called) is what we get when we join up the triangle. This is called the *triangle law* for adding vectors.

There is another way of adding two vectors. Instead of making the second vector start where the first one finishes, we make them both start at the same place, and complete a parallelogram.

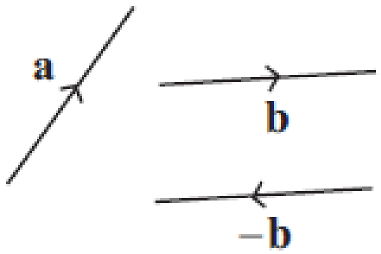
This is called the *parallelogram law* for adding vectors. It gives the same



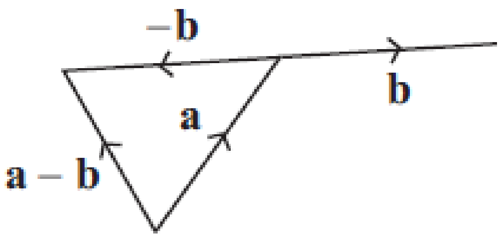
result as the triangle law, because one of the properties of a parallelogram is that opposite sides are equal and in the same direction, so that b is repeated at the top of the parallelogram.

0.03.06 Subtracting two vectors

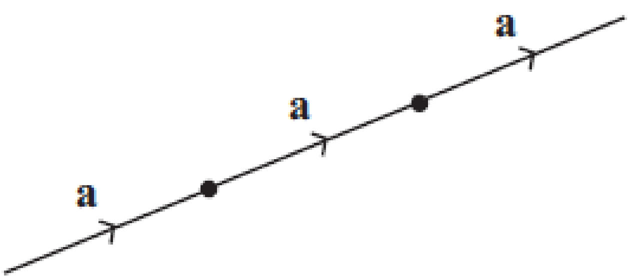
What is a −b? We think of this as a +(−b), and then we ask what −b might mean. This will
be a vector equal in magnitude to b, but in the reverse direction.



Now we can subtract two vectors. Subtracting b from a will be the same as *adding* −b to a.



Adding a vector to itself



What happens when you add a vector to itself, perhaps several times? We write, for example,
a + a + a = 3a.
In same we can write na = a + a+ a ….. n times

Or we can multiply any vector by constant (say n) and result is again a vector having magnitude n times of the previous.

Exercises 0.3.01

In Δ*OAB*, *OA* = a and *OB* = b. In terms of a and b,

(a) What is *AB*?

(b) What is *BA*?

(c) What is *OP*, where *P* is the midpoint of *AB*?

(d) What is *AP*?

(e) What is *BP*?

(f) What is *OQ*, where *Q* divides *AB* in the ratio 2:3?

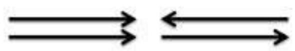
Ans:

(a) b − a (b) a − b (c) $\frac{1}{2}$ (a + b)

(d) $\frac{1}{2}$ (b − a) (e) $\frac{1}{2}$ (a − b) (f) $\frac{3}{5}$ a + $\frac{2}{5}$ b

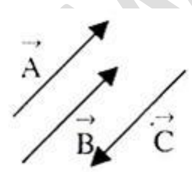
0.03.07 Different types of vectors

1) Collinear vectors:- Vectors having a common line of action are called collinear vectors. There are two types of collinear vectors. One is parallel vector and another is anti-parallel vector.



2) Parallel Vectors:- Two or more vectors (which may have different magnitudes) are said to be parallel (θ = 0°) when they are parallel to the same line. In the figure below, the vectors $\vec{A}$ and $\vec{B}$are parallel.

3) Anti Parallel Vectors:-



Two or more vectors (which may have different magnitudes) acting along opposite direction are called anti-parallel vectors. In the figure below, the vectors $\vec{B}$ and $\vec{C}$ are anti-parallel vectors.

4) Equal Vectors: - Two or more, vectors are equal if they have the same magnitude (length) and direction, whatever their initial points. In the figure above, the vectors A and B are equal vectors.

5) Negative Vectors: - Two vectors which have same magnitude (length) but their direction is opposite to each, other called the negative vectors of each other. In figure above vectors A and C or B and C are negative vectors.

6) Null Vectors: - A vector having zero magnitude an arbitrary direction is called zero vector or 'null vector' and is written as = O vector. The initial point and the end point of such a vector coincide so that its direction is indeterminate. The concept of null vector is hypothetical but we introduce it only to explain some mathematical results.

Properties of a null vector:-

(a) It has zero magnitude.

(b) It has arbitrary direction

(c) It is represented by a point.

(d) When a null vector is added or subtracted from a given vector the resultant vector is same as the given vector.

(e) Dot product of a null vector with any vector is always zero.

(f) Cross product of a null vector with any vector is also a null vector.

Co-planar Vector: - Vectors situated in one plane, irrespective of their directions, are known as co-planar vectors.

0.03.08 Unit Vector or vector of unit length

If a is vector and |a| represents the magnitude of vector then $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Represent unit vector. Unit vector $\hat{a}$ is called as a-cap or a-hat.

Thus if vector a has magnitude of p then $\vec{a} = p\,\hat{a}$

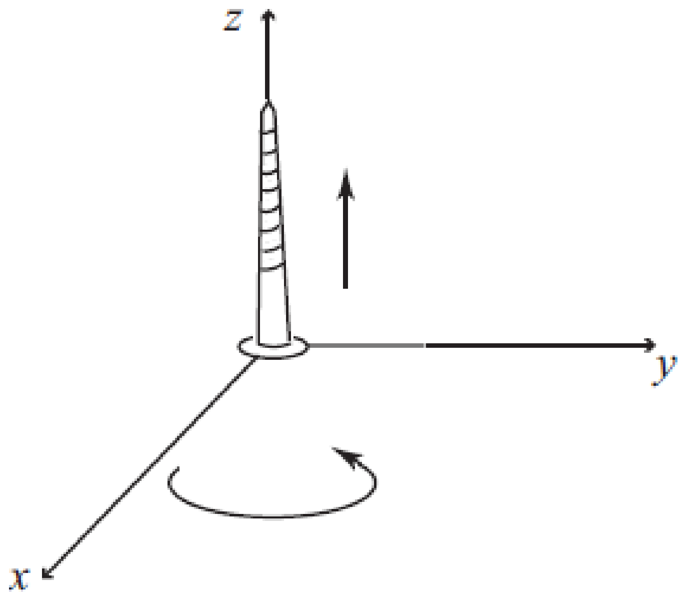
Unit vector along x-axis is represent by $\hat{\imath}$

Unit vector along y-axis is represent by $\hat{\jmath}$

Unit vector along z-axis is represent by $\hat{k}$

0.03.09 Cartesian frame of reference

In three dimensions we have three axes, traditionally labeled x, y and z, all at right angles to each other. Any point P can now be described by three numbers, the coordinates with respect to the three axes.



Now there might be other ways of labeling the axes. For instance we might interchange x and y, or interchange y and z. But the labeling in the diagram is a standard one, and it is called a right-handed system. Imagine a right-handed screw, pointing along the z-axis. If you tighten the screw, by turning it from the positive x-axis towards the positive y-axis, then the screw will move along the z-axis. The standard system of labeling is that the direction of movement of the screw should be the positive z direction.

This works whichever axis we choose to start with, so long as we go round the cycle x, y, z, and then back to x again. For instance, if we start with the positive y-axis, then turn the screw towards the positive z-axis, then we'll tighten the screw in the direction of the positive x-axis.

Algebraic representation of vector

0.03.10 Vectors in two dimensions.

The natural way to describe the position of any point is to use Cartesian coordinates. In two dimensions, we have a diagram like this, with an x-axis and a y-axis, and an origin O. To include vectors in this diagram, we have unit vector along x-axis is denoted by $\hat{\imath}$ and a unit vector along y-axis is denoted by $\hat{\jmath}$. In figure (a) vector along positive x-axis having magnitude of 6 is represented as $6\hat{\imath}$, while in figure (b) vector along positive y axis having magnitude is represented as $5\hat{\jmath}$.

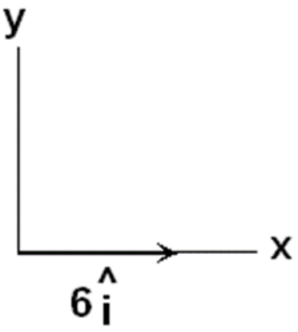


figure (a)

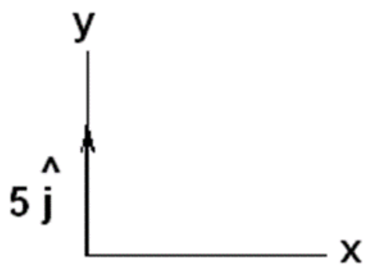
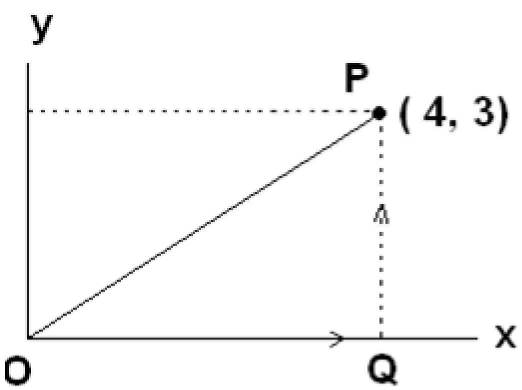


figure (b)



$\vec{R} = \vec{OP} = 4\hat{i} + 3\hat{j}$

In adjacent figure point P has co-ordinates (4, 3). If we want to reach from point O to P. We can walk 4 units along positive x-axis and 3 units after taking 90° turn.

Thus vector $\overrightarrow{OP}$ is result of the addition of two vectors $\overrightarrow{OQ}$ and $\overrightarrow{QP}$. Mathematically it can be represented as

$$\overrightarrow{OP} = 4\hat{\imath} + 3\hat{\jmath}$$

$4\hat{\imath}$ Can be said to be x-component of vector OP. or effectivity of vector OP along x-axis is 4 unit.

$3\hat{\jmath}$ Can be said to be y-component of vector OP or effectivity of vector OP along y-axis is 3 unit.

Angle made by the vector with x-axis

$$tan\theta = \frac{3}{4}$$

In general

$$tan\theta = \frac{y - component}{x - component}$$

Example : If force of $\vec{R} = (6\hat{\imath} + 8\hat{\jmath})$ N then force 6N force will cause motion along positive x axis and 8 N force will cause motion along y –axis.

If mass of object is 2kg. Then acceleration along +ve x-axis will be 6/2 = 3 m/s^2 and acceleration along +ve y-axis will be 8/2 = 4 m/s^2.

Magnitude of vector R can be calculated using Pythagoras equation

$$\vec{R} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$$

And unit vector is $\hat{R} = \frac{\vec{R}}{|\vec{R}|}$

$$\hat{R} = \frac{6\hat{\imath} + 8\hat{\jmath}}{10} = \left(\frac{6}{10}\hat{\imath} + \frac{8}{10}\hat{\jmath}\right) unit$$

Vector R can be represented in terms of unit vector as magnitude × unit vector

$$\vec{R} = 10\left(\frac{6}{10}\hat{\imath} + \frac{8}{10}\hat{\jmath}\right) unit$$

In general vector in two dimension is represented as

$$\vec{A} = x\hat{\imath} + y\hat{\jmath}$$

Magnitude as

$$|\vec{A}| = \sqrt{x^2 + y^2}$$

Unit vector

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|}$$

Angle made by force vector

$$tan\theta = \frac{y - component}{x - component} = \frac{8}{6} = 1.333$$

θ=53^o 8′

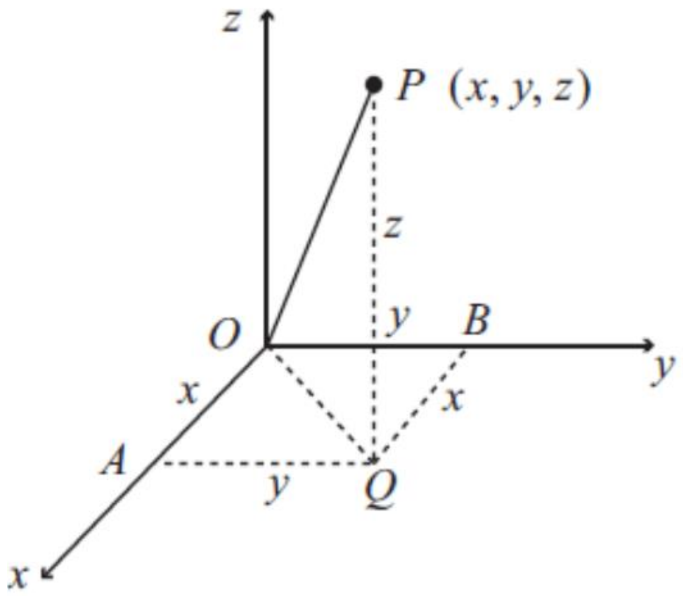
0.03.11 Vectors in three dimensions.

Vector OP in adjoining figure represent vectors in three dimensions and can be represented as

$$\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{\jmath}$$

And magnitude as

$$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$$



Unit vector is

$$\widehat{OP} = \frac{\overrightarrow{OP}}{|\overrightarrow{OP}|}$$

x, y and z are component of vector along x-axis , y axis and z-axis.

Algebraic addition of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Both vectors can be added to get resultant vector

$$\vec{R} = \vec{A} + \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) + (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$\vec{R} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$

If $$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}$$

From above we get

$R_x = A_x + B_x$, $R_y = A_x + B_x$, $R_z = A_z + B_z$

Algebraic subtraction of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Both vectors can be added to get resultant vector

$$\vec{R} = \vec{A} - \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}) - (B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$\vec{R} = (A_x - B_x)\hat{\imath} + (A_y - B_y)\hat{\jmath} + (A_z - B_z)\hat{k}$$

If

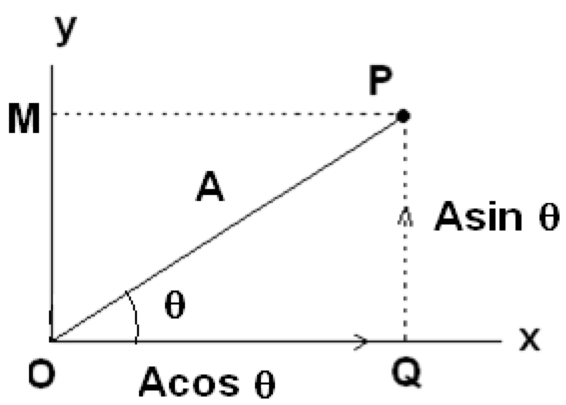
$$\vec{R} = R_x\hat{\imath} + R_y\hat{\jmath} + R_z\hat{k}$$

From above we get

$$R_x = A_x - B_x,\ R_y = A_x - B_x\ ,\ R_z = A_z - B_z$$

Note: when two vectors are added or subtracted, their components get add or subtracted to give new vector

0.03.12 Polar representation of vector



Let vector OP makes an angle of θ with positive x-axis. Then draw a perpendicular from point P on x-axis intercept at Q and perpendicular on y-axis intercept at point M. Then OQ is called projection of vector OP on x-axis

From trigonometry OQ = A cos θ or effectively of OP along x-axis or a component of Vector OP along x-axis

Thus Vector OQ can be represented as $A\,cos\theta\ \hat{\imath}$

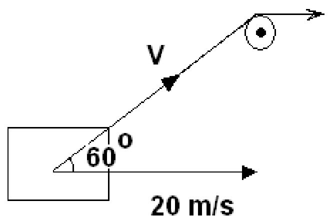
Similarly OM is projection of vector OP on y-axis. From trigonometry MO = PQ = A sin θ or effectively of OP along y-axis or a component of vector along –y-axis. Thus vector QP can be represented as $Asin\theta\hat{\jmath}$

As vector OP is made up of two mutually perpendicular vectors we can get

$$\overrightarrow{OP} = Acos\theta\ \hat{\imath} + Asin\theta\hat{\jmath}$$

Note that is the magnitude of component along OQ or x-axis is x then magnitude of vector will be |A| = x/cos θ

Example :

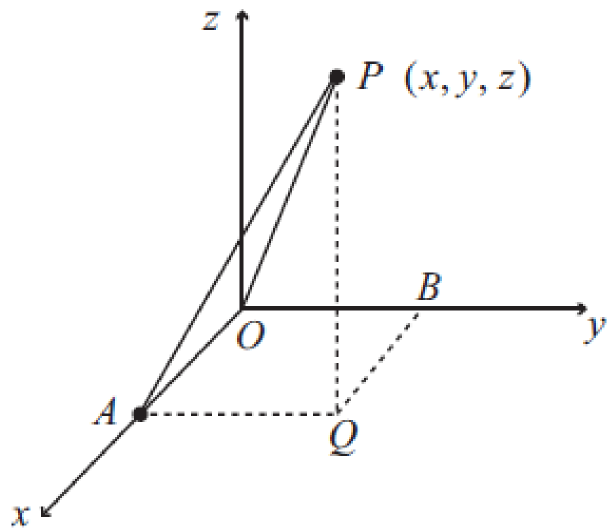


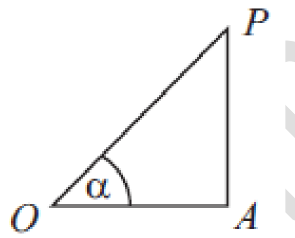
An object as shown in figure is moved with velocity along x-axis is 20m/s. By a thread making an angle of 60^O passing over a pulley. What is the speed of thread V

Solution

Given Vcos60 = 20 ∴ V = 20/Cos60 = 40m/s

Cosine rule





Let vector OP makes an angle of $\alpha$with x –axis, β with y-axis, γ with z-axis.

Since coordinates of 'P' are (x,y,z) and is position vector thus magnitude is

$|\overrightarrow{OP}| = \sqrt{x^2 + y^2 + z^2}$

AP is perpendicular on x axis thus OA = x

But the triangle *POA* is a right-angled triangle, so we can write down the cosine of the angle *POA*. If we call this angle α, then

As shown in figure on right

$$cos\alpha = \frac{OA}{OP} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

The quantity cos α is known as a *direction cosine*, because it is the cosine of an angle which helps to specify the direction of *P*; α is the angle that the position vector *OP* makes with the *x-axis.*

Similarly

$$cos\beta = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$cos\gamma = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

We can also be proved that

$$cos^2\alpha + cos^2\beta + cos^2\gamma = 1$$

Exercises 0.3.02

1. Find the lengths of each of the following vectors

(a) $2\hat{\imath} + 4\hat{\jmath} + 3\hat{k}$ (b) $5\hat{\imath} - 2\hat{\jmath} + \hat{k}$ (c) $2\hat{\imath} - \hat{k}$

(d) $5\hat{\imath}$ (e) $3\hat{\imath} - 2\hat{\jmath} - \hat{k}$ (f) $\hat{\imath} + \hat{\jmath} + \hat{k}$

2. Find the angles giving the direction cosines of the vectors in Question

3. Determine the vector *AB* for each of the following pairs of points

(a) A (3,7,2) and B (9,12,5) (b) A (4,1,0) and B (3,4,-2)

(c) A (9,3,-2) and B (1,3,4) (d) A (0,1,2) and B (-2,1,2)

(e) A (4,3,2) and B (10,9,8)

4. For each of the vectors found in Question 3, determine a unit vector in the direction of) $\overrightarrow{AB}$

Ans:

1. (a)√29 (b)√30 (c)√5 (d) 5 (e)√14 (f)√3

2. (a) 68.2°, 42 °10′ , 56 ° *15′* (b) 24.1 °, 111.4 °, 79.5 °

(c), 26.6 °, 90 ° ,116.6 ° (d) 0 °, 90 ° , 90 °

(e) 36.7 °, 122.3 °, 105.5 ° (f) 54.7 °, 54.7 °, 54.7 °

3. (a) $6\hat{\imath} + 5\hat{\jmath} + 3\hat{k}$ (b) $-\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ (c) $-8\hat{\imath} + 6\hat{k}$

(d) $-2\hat{\imath}$ (e) $6\hat{\imath} + 6\hat{\jmath} + 6\hat{k}$

4. (a) $\frac{1}{\sqrt{70}}(6\hat{\imath} + 5\hat{\jmath} + 3\hat{k})$ (b) $\frac{1}{\sqrt{14}}(-\hat{\imath} + 3\hat{\jmath} - 2\hat{k})$

(c) $\frac{1}{10}(-8\hat{\imath} + 6\hat{k})$ (d) $-\hat{\imath}$ (e) $\frac{1}{\sqrt{3}}(\hat{\imath} + \hat{\jmath} + \hat{k})$

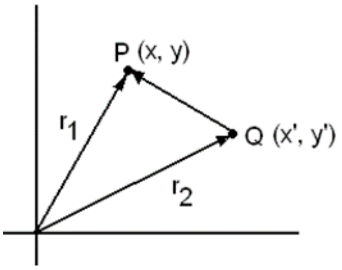
0.03.13 Displacement vector

As shown in figure point P coordinates are (x ,y) while point Q coordinates are (x′, y′)

$$\overrightarrow{r_1} = x\hat{\imath} + y\hat{\jmath}$$

And

$$\overrightarrow{r_2} = x'\hat{\imath} + y'\hat{\jmath}$$



Now Vector QP = $r_1 - r_2$ (*triangle law)*

$$\overrightarrow{QP} = x\hat{\imath} + y\hat{\jmath} - x'\hat{\imath} - y'\hat{\jmath}$$

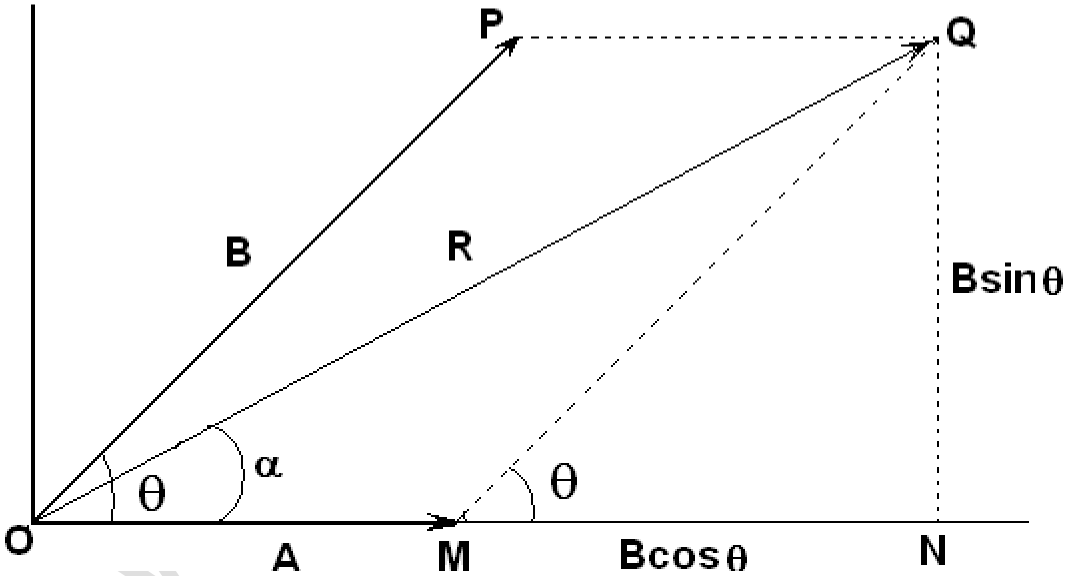
$$\overrightarrow{QP} = x\hat{\imath} - x'\hat{\imath} + y\hat{\jmath} - y'\hat{\jmath}$$

$$\overrightarrow{QP} = (x - x')\hat{\imath} + (y - y')\hat{\jmath}$$

0.03.14 Important result

If A and B are the two vector and θ is the angle between them then we can find a formula to get magnitude and orientation or direction of resultant vector.

We can obtain formula as follows



Let vector A be along x-axis and Vector B is making angle of θ with x-axis. Vector R is the resultant vector according to parallelogram law.

From figure Component of R along x-axis is ON and along y-axis is NQ

$$\vec{R} = \overrightarrow{ON} + \overrightarrow{NQ}$$

But

$$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$$

From the figure OM = A and MN = Bcosθ and MN =Bsinθ

$$\vec{R} = A\hat{\imath} + Bcos\theta\hat{\imath} + Bsin\theta\hat{\jmath}$$

$$\vec{R} = (A + Bcos\theta)\hat{\imath} + Bsin\theta\hat{\jmath}$$

$$\left|\vec{R}\right|^2 = (A + Bcos\theta)^2 + (Bsin\theta)^2$$

$$\left|\vec{R}\right|^2 = A^2 + 2ABcos\theta + B^2cos^2\theta + B^2sin^2\theta$$

$$\left|\vec{R}\right|^2 = A^2 + 2ABcos\theta + B^2$$

$$\left|\vec{R}\right| = \sqrt{A^2 + B^2 + 2ABcos\theta}$$

Orientation or direction of resultant vector from triangle OQN

$$tan\alpha = \frac{QN}{ON} = \frac{QN}{OM + MN}$$

$$tan\alpha = \frac{Bsin\theta}{A + Bcos\theta}$$

Exercises 0.3.03

Find magnitude of resultant vector and orientation

a) |A| = 6 and |B| = 8 θ = 30^O b) |A| = 15 and |B| = 8 θ = 60^O

c) |A| = 12 and |B| = 8 θ = 90^O

Answers

13.53 units , tanα = 0.3095 angle is with vector A

20.22 units , tanα =0.3646 angle is with vector A

14.42 units, tanα = 0.666 angle is with vector A

0.03.15 Scalar product of vectors

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Scalar product or dot product is defined as

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left|\vec{B}\right|cos\theta$$

Not that right hand side of the equation is scalar quantity while right hand side is vector quantity.

θ is the angle between vector A and B . | A | and | B | are the magnitude of vector A and B.

We may right above equation in different form to get more information

$$\vec{A} \cdot \vec{B} = \left|\vec{A}\right|\left(\left|\vec{B}\right|cos\theta\right)$$

Bcosθ is projection of vector B along vector A

$$\vec{A}\cdot\vec{B} = |\vec{A}|(\text{projection of vector B along vector A})$$

Or

$$\vec{A}\cdot\vec{B} = |\vec{B}|(\text{projection of vector A along vector B})$$

If we take dot product of unit vectors

$\hat{\imath}\cdot\hat{\imath} = 1\times 1 cos0 = 1$ as magnitude of unit vector is 1 and angle between two $\hat{\imath}$ is 0,

Similarly $\hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$

$\hat{\imath}\cdot\hat{\jmath} = 1\times 1 cos90 = 0$ as angle between x-axis and y-axis is 90º

Similarly $\hat{\imath}\cdot\hat{k} = \hat{\jmath}\cdot\hat{k} = 0$

Thus we can say that if unit vectors are parallel their dot product is 1. If unit vectors are perpendicular their dot product is zero.

$$\vec{A}\cdot\vec{B} = \left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)\cdot\left(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}\right)$$

$$\vec{A}\cdot\vec{B} = \left(A_xB_x\hat{\imath}\cdot\hat{\imath} + A_xB_y\hat{\imath}\cdot\hat{\jmath} + A_xB_z\hat{\imath}\cdot\hat{k}\right) + \left(A_yB_x\hat{\jmath}\cdot\hat{\imath} + A_yB_y\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\hat{\jmath}\cdot\hat{k}\right) + \left(A_zB_x\hat{k}\cdot\hat{\imath} + A_zB_y\hat{k}\cdot\hat{\jmath} + A_zB_z\hat{k}\cdot\hat{k}\right)$$

$\hat{\imath}\cdot\hat{\imath} = \hat{\jmath}\cdot\hat{\jmath} = \hat{k}\cdot\hat{k} = 1$ and remaining all unit vector dot products give zero value thus

$$\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Uses of dot products

Dot product is used to check vectors are perpendicular or not

To find angle between two vectors

Projection of one vector along the another vector

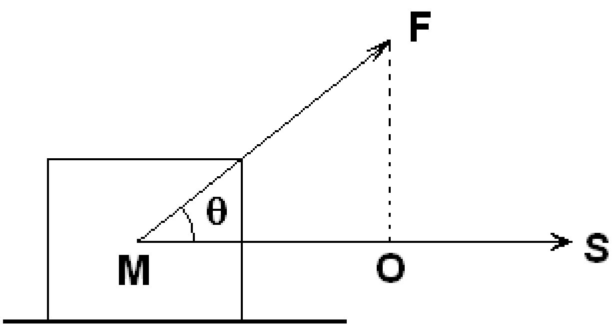
Explanation for why dot product gives scalar

According to definition of work,

work = displacement × force in the direction of displacement

Note work is scalar quantity

Consider following diagram in which force is vectors making angle of θ with displacement which is a vector.



MO is the effective force along MS which represents displacement.

MO = Fcosθ

Now according to definition of work

W= S × Fcosθ

Which can be represented as $W = \vec{S} \cdot \vec{F}$ Thus dot product gives scalar product

Exercises 0.3.04

1. If a = 4i + 9j and b = 3i + 2j find (a) a · b (b) b · a (c) a · a (d) b · b.
2. Find the scalar product of the vectors 5i and 8j.
3. If p = 4i + 3j + 2k and q = 2i − j + 11k find
(a) p · q, (b) q · p, (c) p · p, (d) q · q.
4. If r = 3i + 2j + 8k show that r · r = $|r|^2$.

5 Determine whether or not the vectors 2i + 4j and −i + 0.5 j are perpendicular.

6. Evaluate p · i where p = 4i + 8j. Hence find the angle that p makes with the x axis.
7. Obtain the component of a vector A = 3i+4j in the direction of 2i+2j

Answers

1. (a) 30, (b) 30, (c) 97, (d) 13.
2. 0.
3. (a) 27, (b) 27, (c) 29, (d) 126.
4. Both equal 77.

5 Their scalar product is zero. They are non-zero vectors. We deduce that they must be perpendicular.

6. p · i = 4. The required angle is 63.4°. 7. $\frac{7}{\sqrt{2}}$

0.03.16 Vector product or cross product

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

Vector product or cross product is defined as

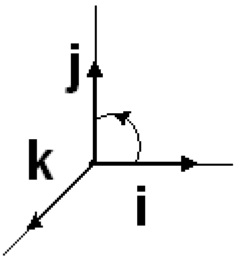
$$\vec{A}\times\vec{B} = |\vec{A}||\vec{B}|sin\theta\hat{n}$$

Note that cross product gives again vector, direction of resultant vector is determined by right hand screw rule.

If we take cross product of unit vectors

$\hat{\imath}\times\hat{\imath} = 1\times1sin0 = 0$ As magnitude of unit vector is 1 and angle between two $\hat{\imath}$ is 0,

Similarly $\hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$



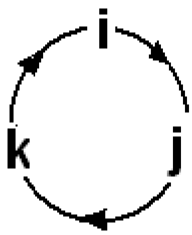
$$\hat{\imath}\times\hat{\jmath} = 1\times1sin90 = \hat{k}$$

As angle between x-axis and y-axis is 90°. Direction of resultant can be obtained by rotating right hand screw in the direction as shown in figure which gives the direction in +z axis

Similarly $\hat{\jmath}\times\hat{k} = \hat{\imath}$ and $\hat{k}\times\hat{\imath} = \hat{\jmath}$

But $\hat{\jmath}\times\hat{\imath} = -\hat{k}$

$\hat{k}\times\hat{\jmath} = -\hat{\imath}$ and $\hat{\imath}\times\hat{k} = -\hat{\jmath}$



This sequence can be remembered by following the arrows for positive resultant vector in adjacent figure and if followed in opposite to direction we get Negative vector

Thus we can say that if unit vectors are parallel their cross product is 0.

$$\vec{A}\cdot\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})\times(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$\vec{A}\times\vec{B} = (A_xB_x\hat{\imath}\times\hat{\imath} + A_xB_y\,\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k}) \quad + (A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_y\hat{\jmath}\times\hat{\jmath} + A_yB_z\hat{\jmath}\times\hat{k})$$

$$+(A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath} + A_zB_z\hat{k}\times\hat{k})$$

Since $\hat{\imath}\times\hat{\imath} = \hat{\jmath}\times\hat{\jmath} = \hat{k}\times\hat{k} = 0$ above equation reduce to

$$\vec{A}\times\vec{B} = \left(A_xB_y\,\hat{\imath}\times\hat{\jmath} + A_xB_z\hat{\imath}\times\hat{k}\right) + \left(A_yB_x\hat{\jmath}\times\hat{\imath} + A_yB_z\hat{\jmath}\times\hat{k}\right) + \left(A_zB_x\hat{k}\times\hat{\imath} + A_zB_y\hat{k}\times\hat{\jmath}\right)$$

Now $\hat{\imath}\times\hat{\jmath} = \hat{k}$, $\hat{\imath}\times\hat{k} = -\hat{\jmath}$, $\hat{\jmath}\times\hat{\imath} = -\hat{k}$, $\hat{\jmath}\times\hat{k} = \hat{\imath}$, $\hat{k}\times\hat{\imath} = \hat{\jmath}$, $\hat{k}\times\hat{\jmath} = -\hat{\imath}$

Substituting above values we get

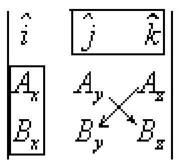
$$\vec{A}\times\vec{B} = \left(A_xB_y\,\hat{k} - A_xB_z\hat{\jmath}\right) + \left(-A_yB_x\hat{k} + A_yB_z\hat{\imath}\right) + \left(A_zB_x\hat{\jmath} - A_zB_y\hat{\imath}\right)$$

By taking common $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ we get

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

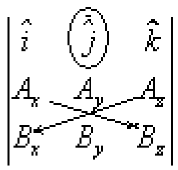
Above equation can be obtained by solving determinant

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



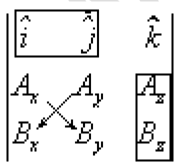
First select $\hat{\imath}$ then follow the arrow as shown in adjacent figure

To get term $\left(A_yB_z - A_zB_y\right)\hat{\imath}$



Similarly select $\hat{\jmath}$ and then follow the arrow as shown in adjacent figure

To get term $\left(A_xB_z - A_zB_x\right)\hat{\jmath}$ give negative sign



select $\hat{k}$ and then follow the arrow as shown in adjacent figure

To get term$\left(A_xB_y - A_yB_x\right)\hat{k}$

Now add all these terms to get equation as

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{\imath} - \left(A_xB_z - A_zB_x\right)\hat{\jmath} + \left(A_xB_y - A_yB_x\right)\hat{k}$$

Note that A × B = - B × A negative sign indicate directions are opposite

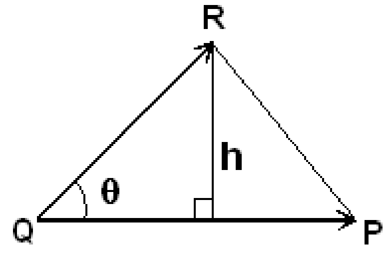
The vector product is distributive over addition. This means

a × (b + c) = a × b + a × c

Equivalently,

(b + c) × a = b × a + c × a

Important results

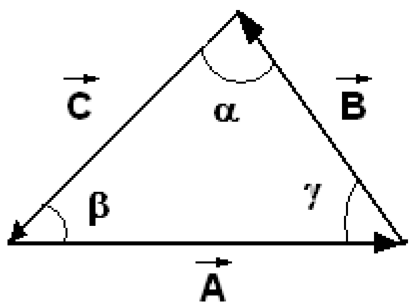


From figure area of triangle QPR =

$$A = \frac{1}{2}|\overrightarrow{QP}|h$$

$$A = \frac{1}{2}|\overrightarrow{QP}||\overrightarrow{QR}|sin\theta$$

$$A = \frac{1}{2}(\overrightarrow{QP}\times\overrightarrow{QR})$$



From above derivation for area we get

$$|\vec{A}\times\vec{B}| = |\vec{B}\times\vec{C}| = |\vec{C}\times\vec{A}|$$

AB sin(180 −γ) = BC sin(180-α) = CA sin(180-β)

Dividing each term by ABC, we have

$$\frac{\sin\gamma}{C} = \frac{\sin\alpha}{A} = \frac{\sin\beta}{B}$$

Exercises 0.3.05

1. Find the cross products of following vectors

(a) p = i + 4j + 9k, q = 2i − k.

(b) p = 3i + j + k, q = i − 2j − 3k.

2. For the vectors p = i + j + k, q = −i − j − k show that, in this special case, p × q = q × p.

3.For the vectors a = i + 2j + 3k, b = 2i + 3j + k, c = 7i + 2j + k, show that

a × (b + c) = (a × b) + (a × c)

4. Find a unit vector which is perpendicular to both a = i + 2j − 3k and b = 2i + 3j + k.

5. Calculate the triple scalar product (a × b) · c when a = 2i − 2j + k, b = 2i + j and c = 3i + 2j + k.

Answers

1.(a) −4i + 19j − 8k, (b) −i + 10j − 7k.

2.Both cross products equal zero, and so, in this special case p × q = q × p. The two given vectors are anti-parallel.

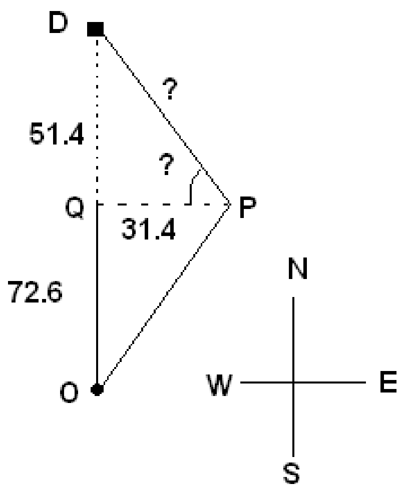
3. Both equal −11i + 25j − 13k

4. $\frac{1}{\sqrt{171}}(11\hat{\imath} - 7\hat{\jmath} - \hat{k})$ 5. 7

Solved numerical

A ship sets out to sail a point 124 km due north. An unexpected storm blows the ship to a point 72.6 km to the north and 31.4 km to the east of its starting point. How far, and in what direction, must it now sail to reach its original destination?

Solution



As shown in figure O is starting point , reached to point P due to wind. , OQ = 72.6 km, Thus QD = 51.4. given QP = 31.4 km.

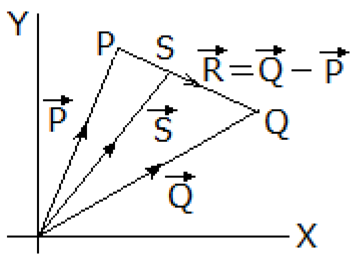
From the triangle DQP $tan\theta = \frac{51.4}{31.4}$

θ = 58.5°

Therefore the ship must go in the direction 58.5° north of west to reach its destination

Using Pythagoras we get PD = 62.2 km

Q) Three vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are shown in the figure. Let S be any point on the vector $\vec{R}$ The distance between the points P and S is $b\,|\vec{R}|$. The general relation among vectors$\vec{P}$, $\vec{Q}$ and $\vec{S}$ is … [IIT advance 2017]



$a)\ \vec{S} = (1-b)\vec{P} + b^2\vec{Q}$

$b)\ \vec{S} = (b-1)\vec{P} + b\vec{Q}$

$c)\ \vec{S} = (1-b)\vec{P} + b\vec{Q}$

d) $\vec{S} = (1 - b^2)\vec{P} + b\vec{Q}$

Solution:

From figure

$\vec{S} = \vec{P} + b\vec{R}$

But $\vec{R} = \vec{Q} - \vec{P}$

$\vec{S} = \vec{P} + b(\vec{Q} - \vec{P})$

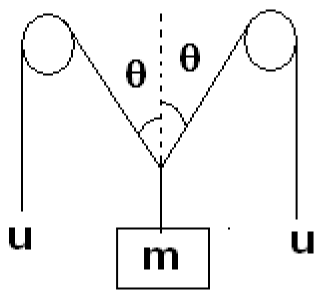
On rearranging terms

$\vec{S} = (1 - b)\vec{P} + b\vec{Q}$

Correct option c

Exercises 0.3.06

1. If the magnitude of vector A, B and C are 12, 5 and 13 units respectively and A+B = C, then what is the angle between A and B
2. The vectors 3i -2j + k and 2i + 6j + ck are perpendicular then find value of c.
3. If A = 2i+3j+k and B = 3i – 2j then what will be dot product.
4. If A= Acosθ i – Asinθj find the vector which is perpendicular to A
5. A vector A of magnitude 10 units and another vector of magnitude 6 units are acting at 60^{O} to each other. What is the magnitude of vector product of two vectors
6. A river is flowing at the rate of 6km/hr. A swimmer across with a velocity of 9 km/hr. What will be the resultant velocity of the man in(km/hr)
7. At what angle two forces 2F and √2 F act, so that the resultant force is F√10
8. In the arrangement shown in figure rope is pulled with velocity u in down



Then at what speed mass m will move up

9. If A · B = AB then what is angle between A and B
10. If A = 2i+3j and B=i+4j+k, then what will be the unit vector along (A+B)
10. A vector x is added to two vectors A = 3i -5j+7k and B = 2i + 4j -3k
11. 0.4 i +0.8 j + ck represents a unit vector when c is
12. A boat moves 10 km due west, 5 km due north, and then 10 km due east. The displacement of the boat from its initial position is
13. Unit vector along i+j is

Answers: 1)180° 2) C = 6

3) 0	4) Bsinθ i + Bcosθ j
5) 30√3 unit	6) √117
7)45°	8) u/cosθ
9) 0	10)$\frac{1}{\sqrt{59}}(3\hat{\imath} + 7\hat{\jmath} + \hat{k})$
11) -5i + 2j -4k	12) √(0.2)
13) 5 km ,North	14) $\frac{\hat{\imath}+\hat{\jmath}}{\sqrt{2}}$

0.04 Differentiation

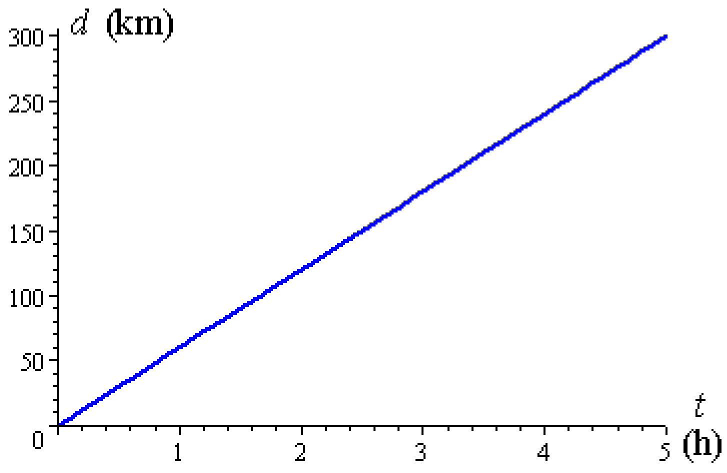
What is Differentiation?

Differentiation is all about finding rates of change of one quantity compared to another. We need differentiation when the rate of change is not constant.

What does this mean?

0.04.01 Constant Rate of Change

First, let's take an example of a car travelling at a constant 60 km/h. The distance-time graph would look like this:



We notice that the distance from the starting point increases at a constant rate of 60 km each hour, so after 5 hours we have travelled 300 km. We notice that the slope (gradient) is always 300/5 =60 for the whole graph. There is a constant rate of change of the distance compared to the time. The slope is positive all the way (the graph goes up as you go left to right along the graph.)

Equation of line is d = 60t, here d is distance and t is time and 60 is slope of the equation and is constant

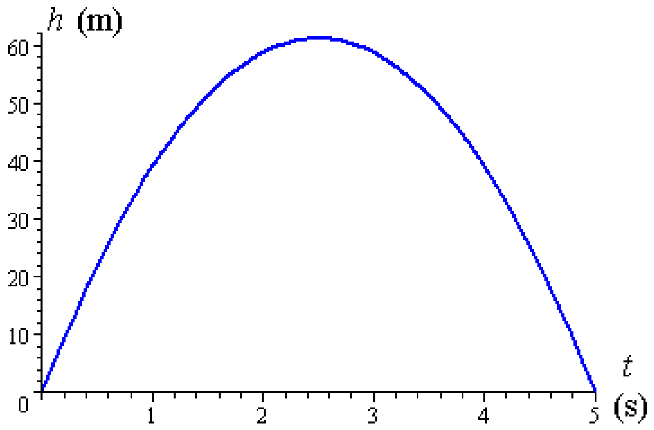
Now if we take a very small period of time then for that very small time displacement will be also very small but ratio of displacement and time will be again 60 km/hr. Such time period which is tending to zero time period but not zero is denoted by dt. Note it is not d and t but one word dt = very small time period tending to zero. And ds denote very small displacement for such very small period.

Now the ratio of ds and dt is known as instantaneous velocity or velocity at that particular time. Similar to velocity noted by us when we look into the Speedo meter. If Speedo meter 60 km/hr it means speed of the car is 60km/hr at that particular moment. $v = \frac{ds}{dt}$

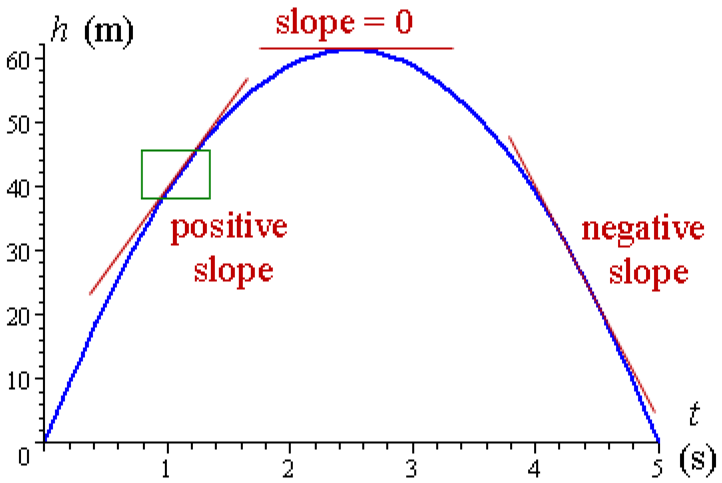
0.04.02 Rate of Change that is Not Constant

Now let's throw a ball straight up in the air. Because gravity acts on the ball it slows down, then it reverses direction and starts to fall. All the time during this motion the velocity is changing. It goes from positive (when the ball is going up), slows down to zero, then becomes negative (as the ball is coming down). During the "up" phase, the ball has negative acceleration and as it falls, the acceleration is positive.

Now let's look at the graph of height (in metres) against time (in seconds).



Notice this time that the slope of the graph is changing throughout the motion. At the beginning, it has a steep positive slope (indicating the large velocity we give it when we throw it). Then, as it slows, the slope get less and less until it becomes 0 (when the ball is at the highest point and the velocity is zero). Then the ball starts to fall and the slope becomes negative (corresponding to the negative velocity) and the slope becomes steeper (as the velocity increases).



TIP :

The slope of a curve at a point tells us the rate of change of the quantity at that point

We can differentiate function, which relates two interdependent variables. For example if position of object is changed from one point to another, time also change with it thus we can have mathematical equation for displacement in terms of time. As shown in first graph equation was S = 60t

For object going up against gravitational force is given by equation of motion is

$$S = ut - \frac{1}{2}gt^2$$

Note here that u is constant, g is also constant. Thus we can easily differentiate above equation to get equation for velocity which can indicate velocity of the object at that particular time. To differentiate above equation we have learn common derivatives of polynomial which are stated below without proof

0.04.03 Derivatives of Polynomials

Common derivatives

a) Derivative of a Constant

$$\frac{dc}{dx} = 0$$

This is basic. In English, it means that if a quantity has a constant value, then the rate of change is zero.

Example

$$\frac{d5}{dx} = 0$$

b) Derivative of n-th power of x

$$\frac{d}{dx}x^n = nx^{n-1}$$

But

$$\frac{d}{dt}x^n = nx^{n-1}\frac{dx}{dt}$$

Example

$$\frac{d}{dx}x^5 = 5x^4$$

c) Derivative of Constant product

$$\frac{d}{dx}(cy) = c\frac{dy}{dx}$$

Exercises 0.4.01

Find the derivative of each of the following with respect to x:

a) x^{6} b) p^{10} c) $\frac{3}{2}x^2$ d) x^{-2} e) $5x^{-5}$ f) $x^{13/2}$ g) $6q^{3/5}$ h) $\frac{2}{3}x^{\frac{3}{2}}$ i) $\frac{1}{x^4}$ j) $\sqrt{x}$ k) $x^{-3/2}$ l) $y^{-1/5}$

Answers : a) $6x^5$ b) $10p^9\frac{dp}{dx}$ c) 3x d) $-2x^{-3}$ e) $-25x^{-6}$ f) $\frac{13}{2}x^{\frac{11}{2}}$

g) $\frac{18}{5}q^{\frac{-2}{5}}\frac{dq}{dx}$ h) $x^{1/2}$ i) $-4x^{-5}$ j) $\frac{1}{2}x^{\frac{-1}{2}}$ k) $\frac{-3}{2}x^{\frac{-5}{2}}$ l) $\frac{-1}{5}y^{\frac{-6}{5}}\frac{dy}{dx}$

0.04.04 Linearity rules

We frequently express physical quantities in terms of variables. Then, functions are used to describe the ways in which these variables change. We now look at some more examples which assume that you already know the following rules: if

y =v ± u , here v and u are functions of 'x'

$$\frac{dy}{dx} = \frac{dv}{dx} \pm \frac{du}{dx}$$

Example 1

Example Suppose we want to differentiate $y = 6x^3 - 12x^4 + 5$.

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3 - 12x^4 + 5)$$

$$\frac{dy}{dx} = \frac{d}{dx}(6x^3) - \frac{d}{dx}(12x^4) + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 6\frac{d}{dx}(x^3) - 12\frac{d}{dx}(x^4) + \frac{d}{dx}(5)$$

$$\frac{dy}{dx} = 6\times3x^2 - 12\times4x^3 + 0$$

$$\frac{dy}{dx} = 18x^2 - 48x^3$$

Example 2: Equation for displacement is given by, find equation for velocity and acceleration

$$S = ut - \frac{1}{2}gt^2$$

$$v = \frac{ds}{dt} = \frac{d}{dt}\left(ut - \frac{1}{2}gt^2\right)$$

$$v = \frac{d}{dt}(ut) - \frac{d}{dt}\left(\frac{1}{2}gt^2\right)$$

Since u and g and ½ are constant

$$v = u\frac{d}{dt}(t) - \frac{1}{2}g\frac{d}{dt}(t^2)$$

$$v = u - \frac{1}{2}g(2t) \;\ldots(i)$$

v = u – gt is the equation for velocity

By taking further derivative or (i) with respect to time we get acceleration

$$a = \frac{dv}{dt} = -g$$

Similarly power is rate of work done per unit time, can be written as

$$P = \frac{dw}{dt}$$

Example: $y = (5x^3+2x)^2$

When we to take derivative of function consists of power of bracket, the first take derivative of bracket, then take derivative of function in the bracket

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 2x)^2$$

$$\frac{dy}{dx} = 2(5x^3 + 2x)^{2-1}\frac{d}{dx}(5x^3 + 2x)$$

$$\frac{dy}{dx} = 2(5x^3 + 2x)^1(15x^2 + 2)$$

Exercises 0.4.02

Find the derivative of each of the following:

a) $X^2 + 12$ b) x^5+x^3+2x

c) $(2x^5+x^2)^3$ d) $\left(\frac{2}{5}x^{\frac{3}{6}} + 7x^{-2}\right)^2$

e) $7p^2 - 5q^3 + 12x + 5$ f) $y = x + \frac{1}{x}$

Answers a) 2x b) $5x^4 + 3x^2 + 2$

c) $3(2x^5+x^2)^2 (10x^4 + 2x)$ d) $2\left(\frac{2}{5}x^{\frac{3}{6}} + 7x^{-2}\right)\left(\frac{1}{5\sqrt{x}} - 14x^{-3}\right)$

e) $14p\frac{dp}{dx} - 15q^2\frac{dq}{dx} + 12$ f) $1 - x^{-2}$

Solved numerical

Q1) The displacement x of a particle moving along a straight line at time t is given by $x=a_0+a_1t+a_2t^2$ The find formula for velocity and acceleration.

Solution: as displacement x is the function of time then derivative with respect to time (t) will give us velocity and derivative of velocity will give us acceleration.

$$v = \frac{dx}{dt} = \frac{d}{dt}(a_0 + a_1t + a_2t^2)$$

$$v = \frac{d}{dt}a_0 + \frac{d}{dt}(a_1t) + \frac{d}{dt}(a_2t^2)$$

$$v = \frac{d}{dt}a_0 + a_1\frac{d}{dt}(t) + a_2\frac{d}{dt}(t^2)$$

Since a_0, a_1, a_2 all are constant we get equation for velocity as

$$v = 0 + a_1 + a_2(2t)$$

$$v = a_1 + 2a_2t$$

By again taking derivative of velocity we get equation for acceleration

$$a = \frac{dv}{dt} = \frac{d}{dt}(a_1 + 2a_2t)$$

$$a = \frac{d}{dt}(a_1) + 2a_2\frac{d}{dt}(t)$$

$$a = 2a_2$$

Q2) A particle moves along a straight line OX. At a time t (in seconds) the distance x(in meters) of the particle from O is given by $x=40 + 12t - t^3$ How long would the particle travel before coming to rest?

Solution: Given equation for displacement is a function of time. When particle comes to rest it means its velocity will become zero. Thus by taking first derivative of displacement equation with respect to time we get formula for velocity

$$v = \frac{dx}{dt} = \frac{d}{dt}(40 + 12t - t^3)$$

$$v = \frac{d}{dt}(40) + \frac{d}{dt}(12t) - \frac{d}{dt}(t^3)$$

$$v = 0 + 12 - 3t^2$$

v= $12-3t^2$

Since object comes to rest v = 0

∴ 0 = $12-3t^2$

∴ t = 2 sec

Thus particle will come to rest after 2 seconds, by substituting t=2 sec in equation for displacement we will displacement of particle before it comes to rest

X = 40 +12 (2) - 2^3

X = 40 + 24 - 8 = 56m

But its initial position t= 0 was

X= 40 +12 (0) –(0)3 = 40 cm

Thus displacement in 2 second = final position – initial position = 56 – 40 = 16 cm

Thus particle will travel 16m before it comes to rest.

Q 3) The relation between time and distance x is $t=\alpha x^2+\beta x$, where α and β are constant. Obtain formula for retardation.

Solution:

Since t depends on displacement we can take derivative of t with respect to x

$$\frac{dt}{dx} = 2\alpha x + \beta$$

We know that $\frac{dx}{dt} = v$

$$\frac{1}{v} = 2\alpha x + \beta$$

$$v = (2\alpha x + \beta)^{-1}$$

Now $a = \frac{dv}{dt}$

$$a = \frac{dv}{dt} = (-1)(2\alpha x + \beta)^{-2}\frac{d}{dt}(2\alpha x + \beta)$$

$$a = (-1)(2\alpha x + \beta)^{-2}\left(2\alpha \frac{dx}{dt}\right)$$

$$a = -2\alpha v^2 v \quad [as\ v = (2\alpha x + \beta)^{-1}]$$
$$a = -2\alpha v^3$$

Exercises 0.4.03

1. A particle moves along a straight line such that its displacement at any time 't' is given by s=(t^3-6t^2+3t+4) meters Find the velocity when the acceleration is zero

Ans V =-9 m/s

2. The displacement of a particle is represented by the following equation s=3t^3 + 7t^2 + 5t + 8 , s is in meters and t in second. What will be the acceleration of particle at t=1 s .

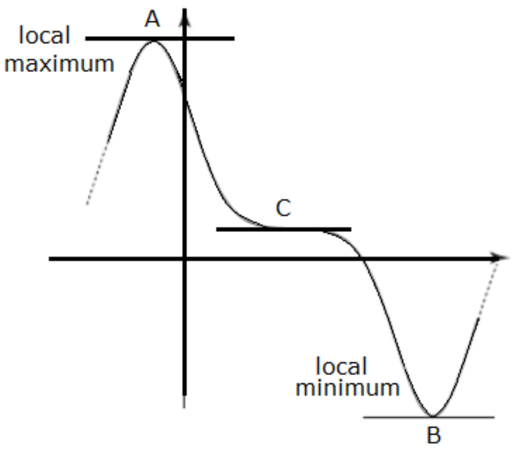
Ans 32 m/s^2

3. The displacement of a particle varies with time(t) as s=at^2 - bt^3.

At what time acceleration of the particle will become zero

Ans $t = \frac{a}{3b}$

0.04.05 Maxima and Minima of function



Refer adjacent graph. Notice that at points A and B the curve actually turns. These two stationary points are referred to as turning points. Point C is not a turning point because, although the graph is flat for a short time, the curve continues to go down as we look from left to right.

So, all turning points are stationary points.

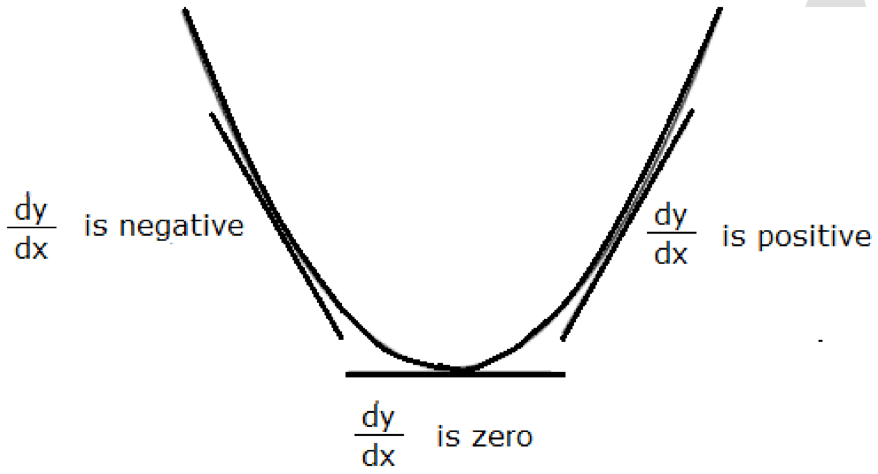
But not all stationary points are turning points (e.g. point C).

In other words, there are points for which $\frac{dy}{dx} = 0$ are stationary point but not necessarily be turning points.

Point A in graph is called a maximum and Point B is called minima

0.04.06 Distinguishing maximum points from minimum points

1) Minimum points



Notice that to the left of the minimum point dy/dx is negative. Because the tangent has negative gradient. At the minimum point, dy/dx is zero. To the right of the minimum point dy/dx is positive because here the tangent has a positive gradient. So, dy/dx goes from negative, to zero, to positive as x increases. In other words, dy/dx must be increasing as x increases.

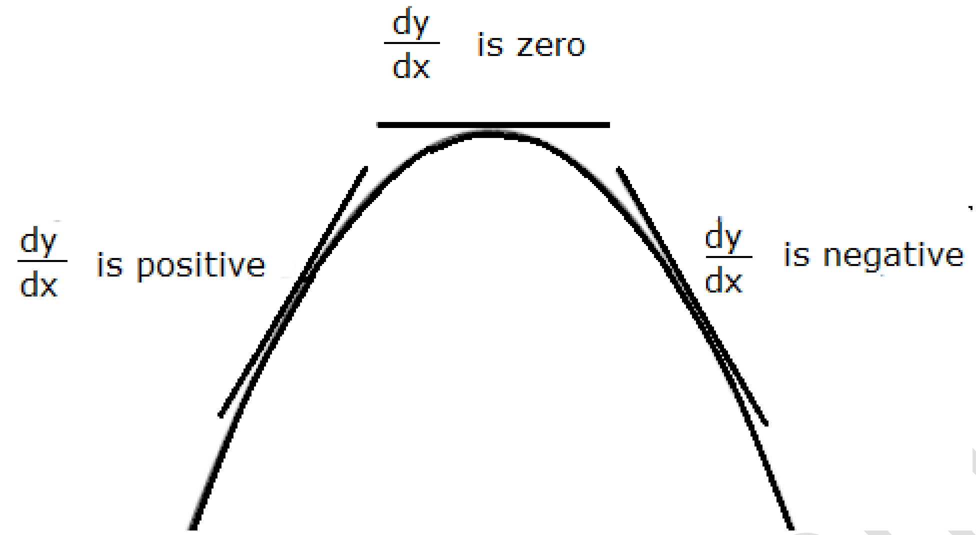
Or rate of change of slope with respect to x is increasing or becomes positive. Thus derivative slope with respect to x is decreasing ,can be represented as

$$\frac{d}{dx}(slope) = \frac{d}{dx}\left(\frac{d}{dx}y\right) = \frac{d^2y}{dx^2}$$

$\frac{d^2y}{dx^2}$ is known as second order derivative .

Note if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$ at a point then that point is minima

2) Maximum points



Notice that to the left of the minimum point, $\frac{dy}{dx}$ is positive because the tangent has positive gradient. At the minimum point, $\frac{dy}{dx} = 0$

To the right of the minimum point $\frac{dy}{dx}$ is negative because here the tangent has a negative gradient. So, $\frac{dy}{dx}$ goes from positive, to zero, to negative

as x increases. In other words, $\frac{dy}{dx}$ must be decreasing as x increases. Or rate of change of slope with respect to x is increasing or becomes negative. Thus derivative slope with respect to x is decreasing

Note if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ at a point then that point is maxima

Example : 1) $y = \frac{1}{2}x^2 - 2x$

$$\frac{dy}{dx} = x - 2 \;\ldots\ldots eq(1)$$

For finding stationary point equate $\frac{dy}{dx} = 0$

X – 2 = 0 Thus x = 2

Now we will take second order derivative of equation to check maxima or minima

$$\frac{d^2y}{dx^2} = -2$$

Since second-order derivative is less than zero, At point x = 2 equation attends maxima and to find the maximum value substitute x = 2 in equation

$$y = \frac{1}{2}x^2 - 2x$$

$$y = \frac{1}{2}2^2 - 2\times 2 = -2$$

Answer is : maximum at (2,−2)

Q. 2) $y = 2x^3 - 9x^2 + 12x$

First order derivative

$$\frac{dy}{dx} = 6x^2 - 18x + 12 \ldots\ldots(1)$$

Equate $6x^2 - 18x + 12 = 0$

Or $x^2 - 3x + 2 = 0$

Root x = +2 and x = +1

To check maxima and minima take derivative of equation 1

$$\frac{d^2y}{dx^2} = 12x - 18 \ldots\ldots(2)$$

Substituting x = +2

We get

$$\frac{d^2y}{dx^2} = 12(2) - 18 = 6 > 0$$

Since value of second order derivative at x = +2 is greater than zero, at point x =+2 function has minima by substituting x = 2 in given equation we get

$y = 2x^3 - 9x^2 + 12x$

$y = 2(2)^3 - 9(2)^2 + 12(2) = 4$

Minimum at (2,4)

Now by Substituting x = +1

We get

$$\frac{d^2y}{dx^2} = 12(1) - 18 = -6 < 0$$

Since value of second order derivative at x = +1 is less than zero, at point x =+1 function has maxima by substituting x = 1 in equation

$y = 2x^3 - 9x^2 + 12x$

$y = 2(1)^3 - 9(1)^2 + 12(1) = 5$

Maximum at (1,5)

Exercises 0.4.04

Locate the position and nature of any turning points of the following functions.

a) $y = x^2 + 4x + 1$ b) $y = 12x - 2x^2$

c) $y = -3x^2 + 3x + 1$ d) $y = x^4 + 2$

e) $y = 7 - 2x^4$ f) $y = 4x^3 - 6x^2 - 72x + 1$

g) $y = -4x^3 + 30x^2 - 48x - 1$,

Ans a) Minimum at (−2,−3)

b) Maximum at (3,18)

c) Maximum at (1/ 2 ,7/4),

d) not Minimum not maximum at (0, 2),

e) not Minimum not maximum at (0, 7),

f) Maximum at (−2, 89), minimum at (3,−161),

g) Maximum at (4, 31), minimum at (1,−23),

Application of maxima and minima to real life problems

Q1) The daily profit, P of an oil refinery is given by $P = 8x - 0.02x^2$, here x is the number of barrels production per day. Find the value of x for which profit become maximum

Solution:

Take derivative and equate it with zero to find value of x

$$\frac{dP}{dx} = 8 - 0.04x$$

8 -0.04x = 0

X = 200

Now verify value of x is for maxima or not by taking second order derivative of P

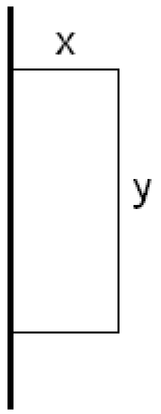
$$\frac{d}{dx}\left(\frac{d}{dx}P\right) = -0.04 < 0$$

As second order derivative gives negative value thus function have maximum value at x = 200

Thus maximum profit P = 8(200) – 0.02(200)^2 = 1600 – 800 = 800 $ per day

Q2) A rectangular storage area is to be constructed alongside of a tall building. A security fence is required along the three remaining side of the area. What is the maximum area that can be enclosed with 800 m of fence wire?

Ans:



Length of fence wire = 2x+ y = 800

Y = 800 – 2x

Area enclosed = xy

Area A = x (800 -2x) = 800x – $2x^2$

To find maxima take first order derivative w.r.t x and equate to zero

$$\frac{dA}{dx} = 800 - 4x$$

800-4x = 0

X =200 m and y = 800 – 2x = 800 – 400 = 400 m

Verification of maxima

$$\frac{d^2A}{dx^2} = -4 < 0$$

Since second order derivative is less than zero for x=200 area is maximum

Area = xy = 200 ×(400) = 80,000 m^2

Maximum area that can be enclosed = 80,000 m^2

Try your self

Q3) A box with a square bas has no top. If 64√3 cm^2 of material is used, what is the maximum possible volume for the box

Some more important rules

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{(v)^2}$$

Some more formula of Differentiation

$$\frac{d}{dx} sinx = cosx$$

$$\frac{d}{dx} cosx = -sinx$$

$$\frac{d}{dx} tanx = sec^2x$$

$$\frac{d}{dx} cotx = -cosec^2x$$

$$\frac{d}{dx} secx = secx\ tanx$$

$$\frac{d}{dx} cosecx = -cosecx\ cotx$$

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x}$$

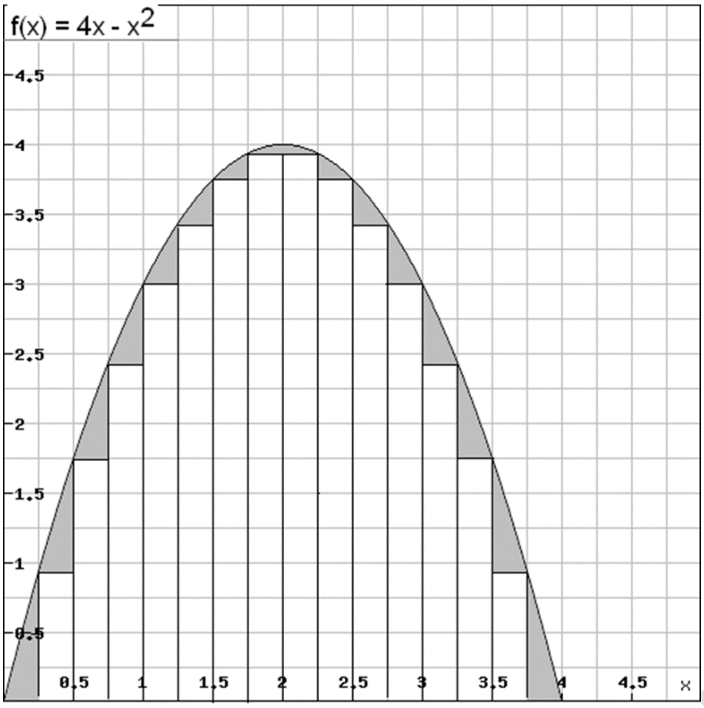
$$\frac{d}{dx}\ a^x = a^x\ log_e a$$

0.05 Integration

0.05.01 Definition

In mathematics, an integral assigns numbers to functions in a way that can describe displacement, area, volume, and other concepts that arise by combining infinitesimal data. Integration is one of the two main operations of calculus, with its inverse, differentiation, being the other.

If we want to calculate the area covered under the graph we have to draw rectangle of fixed width and variable height in the curve.

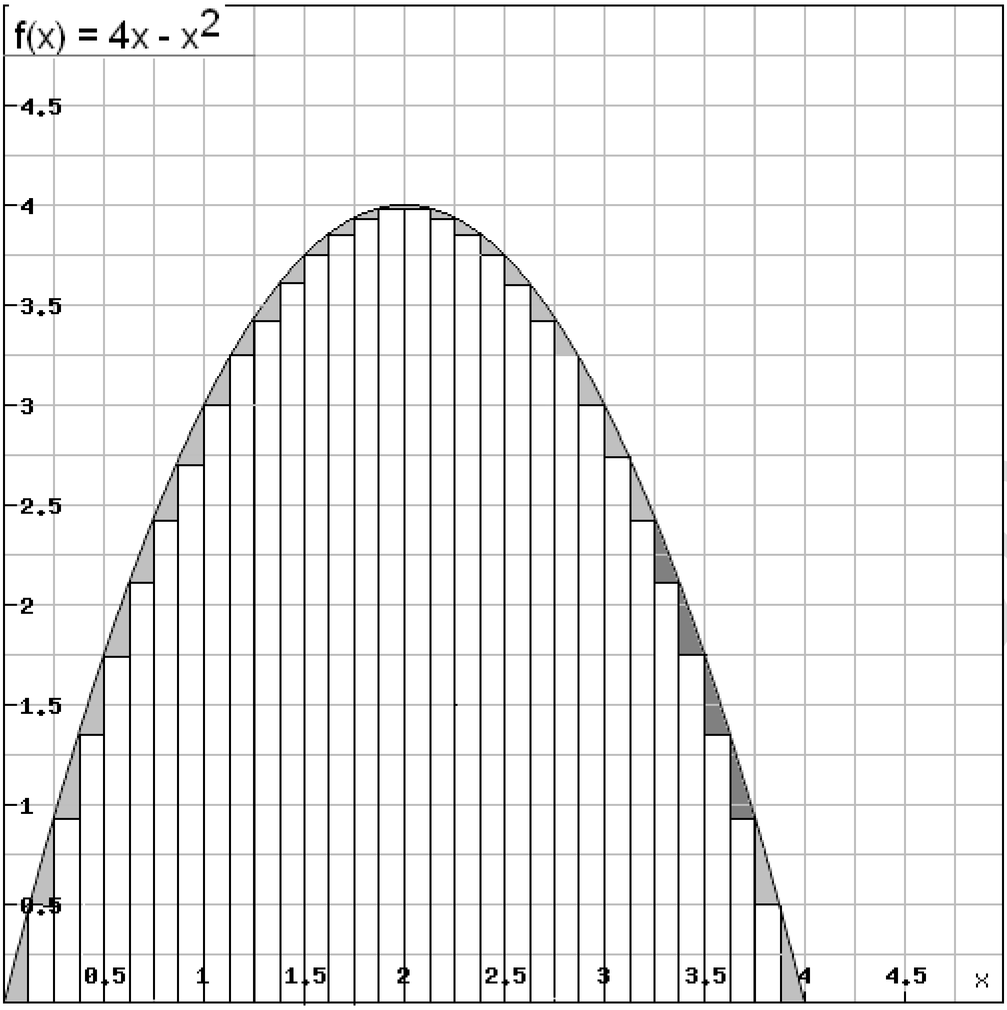


As shown in the graph, we have drawn rectangle of fix with about 0.25. By calculating area of each rectangle and then adding we will get area enclosed by graph much smaller than actual. Area not calculated is shown by shaded portion. Thus to have better approximation of area under curve. We have to take smaller fixed width

If a is the width of each rectangle, and y_1 , y_2, y_3 , … …. are the height of the rectangle than Area under curve A

$A = ay_1 + ay_2 + ay_3 + \ldots\ldots ay_n$

In this case n = 14



As show in figure above now width is taken as 0.125 on x-axis. Then shaded region which we could not calculate by adding area of rectangle is reduced compare to previous 0.25 width

If we continue to take smaller width, shaded region will go on reducing.

If b is smaller width than 'a' then

$A_1 = by_1 + by_2 + by_3 + ... by_n$

In this case n = 30

Clearly $A_1 > A$, and A_1 is more accurate than A

If we make very small with tending to zero, then we will get accurate area under curve

Let dx be very small width. Such that $dx \rightarrow 0$ then are under curve

$A = y_1dx + y_2dx+y_3dx + ...+y_ndx$

Above equation can be expressed mathematically as summation

$$A = \sum_{i=1}^{\infty} y_i \, dx$$

It is not possible for anyone to add millions and trillion of rectangles. Thus summation sign is replaced by ∫ , singe of Summation of function

Now the given graph equation is f(x) = 4x – x^2. Thus our equation for integration reduced to

$$A = \int (4x - x^2) \, dx$$

Above type of integration is called indefinite integral and to not have staring or end point

But in our case starting point is x = 0 called as lower limit and x = 4 called as upper limit our integration equation change to

$$A = \int_0^4 (4x - x^2) dx$$

Without any proof we will use basic formula for integration

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

Here c is called constant of integration

Note in integration power increases by 1 and increased power divides

We cannot integrate above with respect to dt or any other, if function is in terms of y then dy must appear not dx or dt.

Thus by using above formula

$$A = \int_0^4 (4x - x^2) dx$$

$$A = 4\int_0^4 (x) dx - \int_0^7 (x^2) \, dx$$

$$A = 4\left[\frac{x^2}{2}\right]_0^4 - \left[\frac{x^3}{3}\right]_0^4$$

Substitute x= 4 as upper limit and x=0 as lower limit in above equation

$$A = \frac{4}{2}(4^2 - 0^2) - \frac{1}{3}(4^3 - 0^2)$$

$$A = 2(16) - \frac{1}{3}(64)$$

$$A = 32 - 21.33$$

Thus area under the given curve = 10.67 units (accurate up to two decimal)

Exercises 0.5.01

Integrate following

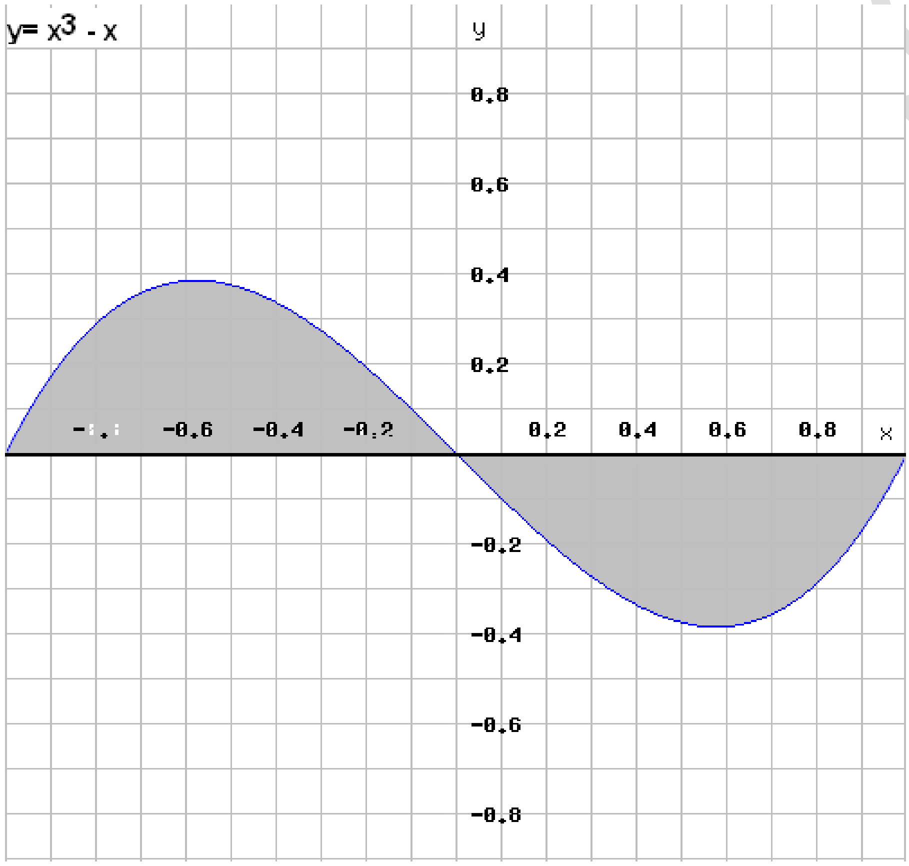
a) $X^{-3/5}$ b) $x^{5/3}$ c) x^{7} d) $X^6 -10x^3 +5$

Answers: a) $\frac{5}{2}x^{\frac{2}{5}}$ b) $\frac{3}{8}x^{8/3}$ c) $\frac{x^8}{8}$ d) $\frac{1}{7}x^7 - \frac{5}{2}x^4 + 6x$

Example

Find the area contained by the curve y = x(x − 1)(x + 1) and the x-axis.

Above function intercept x axis at x= 0



To find other intercept take

$$x(x - 1)(x + 1) = 0$$

(x-1)(x+1) = 0

Thus at x = +1 and x = -1 curve intercept x-axis

Thus we get two regions x=-1 to x = 0 and x =0 to x =1 as shown in graph.

Area enclosed by y=x^3 – x and axis is shown in graph by shaded region.

Therefore we have to integrate above function for two limits

Y =x(x-1)(x+1)

$$Y = x^3 - x$$

$$A = \int_{-1}^{0} (x^3 - x)dx + \int_{0}^{1} (x^3 - x)dx$$

$$A = \left\{ \int_{-1}^{0} x^3 dx - \int_{-1}^{0} x dx \right\} + \left\{ \int_{0}^{1} x^3 dx - \int_{0}^{1} x dx \right\}$$

$$A = \left\{ \left[\frac{x^4}{4}\right]_{-1}^{0} - \left[\frac{x^2}{2}\right]_{-1}^{0} \right\} + \left\{ \left[\frac{x^4}{4}\right]_{0}^{1} - \left[\frac{x^2}{2}\right]_{0}^{1} \right\}$$

$$A = \left\{ \left[0 - \frac{(-1)^4}{4}\right] - \left[0 - \frac{(-1)^2}{2}\right] \right\} + \left\{ \left[\frac{(1)^4}{4} - 0\right] - \left[\frac{(1)^2}{2} - 0\right] \right\}$$

$$A = \left\{ -\frac{1}{4} + \frac{1}{2} \right\} + \left\{ \frac{1}{4} - \frac{1}{2} \right\}$$

$$A = \frac{1}{4} + \left\{ -\frac{1}{4} \right\} = 0$$

Since the graph is symmetric about x axis have equal areas above and below x axis. Thus we get area equal to 0 which is not possible

So to solve such we solve the area under the curves separately and their absolute values are added, so we get

A = |1/4| + |-1/4| = 1/2

$$A = \frac{1}{2}\ units$$

Exercises 0.5.02

1. a) Find the area between the curve y = x(x − 3) and the ordinates x = 0 and x = 5.

b) Find the area bounded by the curve $y = x^2 + x + 4$, the x-axis and the ordinates x = 1 and x = 3.

2. Calculate the value of $\int_{-1}^{+1} x(x-1)(x+1)dx$

3. Calculate the value of $\int_0^6 (4x - x^2)dx$

Answers

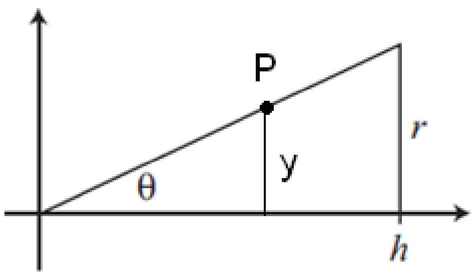
1. a) $\frac{25}{6}\ units$ b) $\frac{62}{3}\ units$

2. -1 /2

3. 0.

0.05.02 Application of integration for finding volume

Volume of Cone



Suppose we have a cone of base radius *r* and vertical height *h*. We can imagine the cone being formed by rotating a straight line through the origin by an angle of 360◦ about the *x*-axis.

If we rotate point around x-axis it will trace a circle of radius y and area πy^2

Now equation for hypo, is y =tan θ x, here tan θ is slope of hypo

Area of circle for radius y, a= $\pi(\tan\theta\ x)^2$

Distance of circles increases from 0 to h and if we add area of all circle = volume of come

$$V = \int_0^h \pi tan^2\theta\, x^2\, dx$$

$$V = \pi tan^2\theta \int_0^h x^2\, dx$$

$$V = \pi tan^2\theta \left[\frac{x^3}{3}\right]_0^h$$

$$V = \pi tan^2\theta \left(\frac{h^3}{3}\right)$$

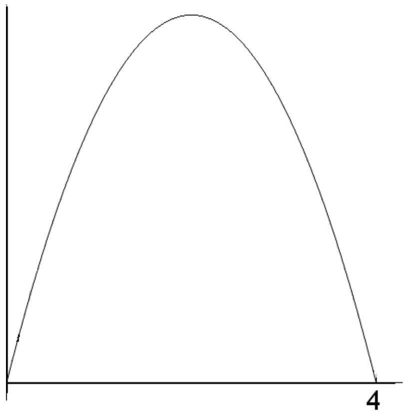
As tanθ =r/h

$$V = \pi tan^2\theta \left(\frac{h^3}{3}\right)$$

$$V = \pi \frac{y^2}{h^2}\left(\frac{h^3}{3}\right)$$

$$V = \pi \frac{hy^2}{3}$$

Example: General equation



Graph shown in figure is for $y = 4x-x^2$

Any point on the graph line if rotated by 360^O around x-axis will represent circle. We can find the volume of the line rotated along x can be calculated as

$V= \int \pi y^2 dx$.

Now upper limit point is x=4 and lower limit point is x=0

$$V = \int_0^4 \pi(4x - x^2)^2\, dx$$

$$V = \int_0^4 \pi(16x^2 - 8x^3 + x^4)\, dx = \pi\left[\left(\frac{16}{3}x^3 - \frac{8}{4}x^4 + \frac{1}{5}x^5\right)\right]_0^4 = 34.1\pi\, units$$

0.05.03 Application of integration in Physics

We can obtain equations of motion using integration for constant acceleration

We know that

1) Acceleration is rate of change of velocity

$$a = \frac{dv}{dt}$$

$$(a)dt = dv \quad ...(1)$$

Let initial velocity be v_0 and final velocity be v at time t then by integrating above equation (1)

$$\int_0^t (a)\, dt = \int_{v_0}^{v} dv$$

$$a[t]_0^t = [v]_{v_0}^{v}$$

$$at = v - v_0$$

2) From the definition of velocity

$$v = \frac{dx}{dt}$$

$$vdt = dx$$

Note Velocity is not constant but acceleration is velocity

$v = v_0 + at$ or $(v_0 + at)\, dt = dx$

Let initial position of object be x_i at time t=0 and final position be x_f at time t

Integrating above equation we get

$$\int_{x_i}^{x_f} dx = \int_0^t (v_0 + at)\, dt$$

On solving above equation we get

$$[x]_{x_i}^{x_f} = \left[v_0 t + a\frac{t^2}{2}\right]_0^t$$

$$x_f - x_i = v_0 t + \frac{1}{2}at^2$$

3) Velocity is changing with time thus must be changing with position

$$\frac{dv}{dx} = ?$$

We will split left side of the above equation

$$\frac{dv}{dt}\times\frac{dt}{dx} = a\times\frac{dt}{dx}$$

$$But\ \frac{dx}{dt} = v \quad \therefore \ \frac{dt}{dx} = \frac{1}{v}$$

$$\frac{dv}{dx} = \frac{a}{v}$$

$$\therefore \quad vdv \ = \ adx$$

Integrating above equation

$$\int_{v_0}^{v}(v)dv = a\int_{x_i}^{x_f} dx$$

$$\left[\frac{v^2}{2}\right]_{v_0}^{v} = a[x]_{x_i}^{x_f}$$

$$\frac{v^2 - v_0^2}{2} = a\left(x_f - x_i\right)$$

$$v^2 - v_0^2 = 2a\left(x_f - x_i\right)$$

$x_f - x_i$ = S (displacement)

Thus by using integration we have proved three basic equations for motion.

Example

Let object starts its motion which was at rest from point P at a distance b from origin under the influence of force given by equation F = $-kx^{-2}$, obtain equation for velocity.

Solution

Equation of force is position dependent

Now F = ma

Ma = $-kx^{-2}$

But

$$a = \frac{dv}{dt}$$

$$\therefore\ m\frac{dv}{dt} = -kx^{-2}$$

$$\frac{dv}{dt} = -\frac{k}{m}x^{-2}$$

$$\frac{dv}{dx}\frac{dx}{dt} = -\frac{k}{m}x^{-2}$$

$$\frac{dv}{dx}v = -\frac{k}{m}x^{-2}$$

$$vdv = -\frac{k}{m}x^{-2}dx$$

Given velocity is zero when particle is at distance b, let v_x be the velocity at distance x from origin

At time t

$$\int_0^{v_x} vdv = -\frac{k}{m}\int_b^x x^{-2}dx$$

$$\frac{v_x^2}{2} = +\frac{k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

$$v_x = \frac{2k}{m}\left(\frac{1}{x}-\frac{1}{b}\right)$$

0.05.04 Some important Integration formulas

$$\int sin\theta\ d\theta = \ -cos\theta$$

$$\int cos\theta\ d\theta = \ sin\theta$$

$$\int tan\theta\ d\theta = \log sec\,\theta = \ -\log cos\theta$$

$$\int cot\theta\ d\theta = \log sin\theta = \ -\log cosec\theta$$

$$\int \frac{1}{x} dx = logx$$

$$\int e^x\, dx = e^x$$

0.06 Quadratic equation

The standard quadratic equation is $ax^2 + bx + c = 0$ where $a \neq 0$.

The solution of the quadratic equation is the values of variable x which satisfied the given quadratic equation.

To solve the quadratic equation factorization is the proper method.

But if polynomial cannot be factorize then to find the roots of equation we find discriminant denoted by Δ or D

$D = b^2 - 4ac$

Depending upon the value of D we have conditions

(i) If $D < 0$, then no real roots for given equation.

(ii) If $D = 0$, then equal and real roots and roots are given by $\frac{-b}{2a}$

(iii) If $D > 0$, then two distinct real roots . The roots are denoted by α and β.

Where

$$\alpha = \frac{-b+\sqrt{D}}{2a}\ and\ \beta = \frac{-b-\sqrt{D}}{2a}$$

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}\ and\ \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Sum of roots = $\alpha + \beta = \frac{-b}{a}$

Product of roots = $\alpha\beta = \frac{c}{a}$

Difference of the roots = $\alpha - \beta = \frac{\sqrt{D}}{a}$

Application of quadratic equation in physics

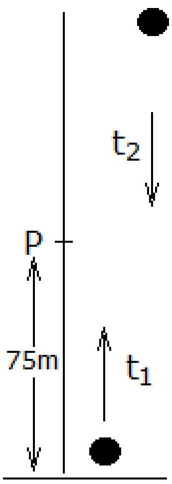
Example

Let object is thrown up with initial speed of 50m/s. At what time it will cross point (P) at height 75 m from ground. And what is the difference between the time when object cross point P while going up and coming down. Take g=10m/s^2

Solution

From equation of motion

$$s = ut + \frac{1}{2}at^2$$



As object is going up and gravitational acceleration down

a=g= -10m/s^2

$$75 = 50(t) - \frac{1}{2}\times 10\times t^2$$

$$t^2 - 10t + 15 = 0$$

Now comparing above equation with standard equation

a =1 , b = -10, c = 15

From formula D = b^2 – 4ac

D= (-10)^2 - 4×1×15

D = 40

As D > 0, two real roots

$$t_2 = \frac{10+\sqrt{40}}{2\times 1} \ \ and \ \ t_1 = \frac{10-\sqrt{40}}{2\times 1}$$

t_2 = 8.162 and t_1 = 1.838

Thus at time 1.838s object will cross point P while going up

And at time 8.162s object will cross point O while going down

Thus difference in timing = 8.162 – 1.838 = 6.234 s

This difference in timing can be calculated directly using formula=

$$\alpha - \beta = \frac{\sqrt{D}}{a}$$

$$t_1 - t_2 = \frac{\sqrt{40}}{1} = 6.324\ s$$

Solve:

a) $x^2 + x - 4 = 0$ b) $x^2 - 3x - 4 = 0$.

c) $6x^2 + 11x - 35 = 0$ d) $x^2 - 48 = 0$.

e) $x^2 - 7x = 0$.

Answer:

a) x = −1, 3 b) x = −1, 4 c) x = −7/2, 5/3 d) x = ± 4√(3) e) x = 0, 7