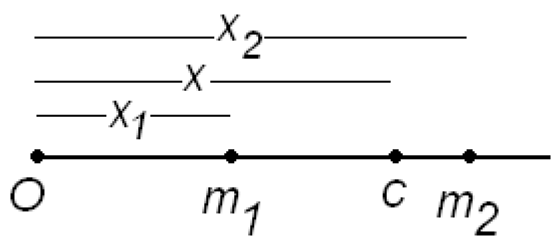
CENTRE OF MASS

As shown in figure consider two particles having mass m_1 and m_2 lying on X-axis at distance of x_1 and x_2 respectively from the origin (O). The centre of mass of this system is that point whose distance from origin O is given by



$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Here, x is the mass-weight average position of x_1 and x_2.

The centre of mass of the two particles of equal mass lies at the centre (on the line joining the two particles between the two particles)

Consider a set of n particles whose masses are m_1 m_2, m_3, … m_n and whose vector relative to an origin O are $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, …….. $\mathbf{r}_n$ respectively

The centre of mass of this set of particles is defined as the point with position vector $\mathbf{r}_{CM}$

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \cdots + m_n\vec{r}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\vec{r}_{CM} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$$

Here M is the total mass of the body.

Solved numerical

Q) Three particles of mass 2kg, 5kg, and 3kg are situated at points with position vectors (i + 4j – 7k)m, (3i – 2j +k)m and (I – 6j +13k) m respectively. Find the position vector of centre of mass

Solution:

$$\vec{r}_{CM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{m_1 + m_2 + m_3}$$

$$\vec{r}_{CM} = \frac{2(\hat{\imath} + 4\hat{\jmath} - 7\hat{k}) + 5(3\hat{\imath} - 2\hat{\jmath} + \hat{k}) + 3(\hat{\imath} - 6\hat{\jmath} + 13\hat{k})}{2 + 5 + 3}$$

$$\vec{r}_{CM} = (2\hat{\imath} - 2\hat{\jmath} + 3\hat{k})\, m$$

Centre of mass of continuous bodies

For calculating centre of mass of continuous body, we first divide the body into suitably chosen infinitesimal elements. The choice is usually determined by the symmetry of the body.

Consider an element dm of the body having position vector r, the quantity $m_i r_i$ can be replaced by $dm\mathbf{r}_i$, direct sum over particles becomes integral over the body

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}dm$$

In component form, this equation can be written as

$$x_{cm} = \frac{1}{M}\int x\, dm$$

$$y_{cm} = \frac{1}{M}\int y\, dm$$

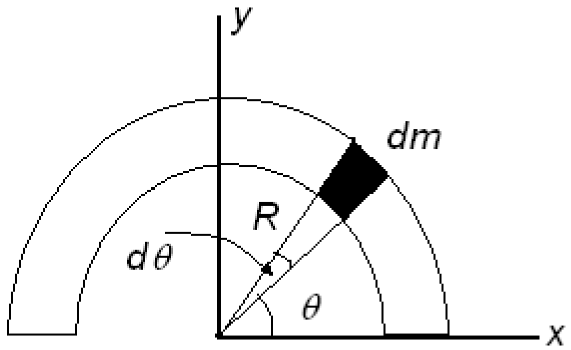
$$z_{cm} = \frac{1}{M}\int z\, dm$$

To evaluate the integral we must express the variable m in terms of spatial coordinates x, y, z or **r**

Solved Numerical

Q) Locate the centre of mass of a uniform semicircular rod of radius R and linear density σ kg/m

Solution



From the symmetry of the body we see at once that the centre of mass of the body must lie along y-axis. So $X_{CM} = 0$.

In this case it is convenient to express the mass element in terms of the angle θ, measured in radian.

The element, which subtends an angle dθ at the origin, has a length Rdθ and a mass dm = σRdθ. Its y coordinate is y = Rsinθ

Therefore

$$y_{CM} = \int_0^{\pi} \frac{y\, dm}{M}$$

$$y_{CM} = \int_0^{\pi} \frac{\sigma R^2 sin\theta d\theta}{M}$$

$$y_{CM} = \frac{\sigma R^2}{M}[-cos\theta]_0^{\pi}$$

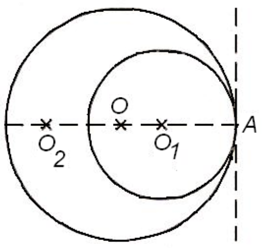
$$y_{CM} = \frac{2\sigma R^2}{M}$$

Total mass of ring M = πRσ

$$\therefore y_{CM} = \frac{2R}{\pi}$$

Q) A circular plate of uniform thickness has a diameter of 56 cm. A circular portion of diameter 42cm is removed from one edge of the plate as shown in figure. Find the centre of mass of the remaining portion

Solution



Let O be the centre of circular plate and O_1, the centre of circular portion removed from the plate. Let O_2 be the centre of mass of the remaining part.

Area of original plate = $\pi R^2 = (28)^2 \pi$ cm^2

Area removed from circular plate = $\pi r^2 = (21)^2 \pi$ cm^2

Let σ be the mass per cm^2. Then

Mass of the original plate m = $(28)^2\pi\sigma$

Mass of the removed part $m_1 = (21)^2 \pi\sigma$

Mass of the remaining part $m_2 = (28)^2\pi\sigma - (21)^2 \pi\sigma = 343\pi\sigma$

Now the masses m_1 and m_2 may be supposed to be concentrated at O_1 and O_2 respectively. Their combined centre of mass is at O. Taking O as origin we have form definition of centre of centre of mass.

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$x_1 = OO_1 = OA - O_1A = 28 - 2\ 1 = 7$cm

$x_2 = OO_2 = ?$, $x_{cm} = 0$

$$0 = \frac{(21)^2\pi\sigma \times 7 + 343\pi\sigma \times x_2}{m_1 + m_2}$$

$$(21)^2\pi\sigma \times 7 + 343\pi\sigma \times x_2 = 0$$

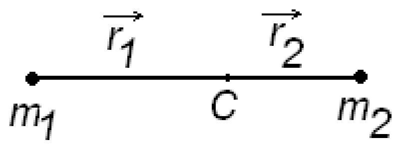
$x_2 = -9$ cm

Q) The distance between two particles of mass m_1 and m_2 is r. If the distances of these particles from the centre of mass of the system are r_1 and r_2 respectively, show that

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right] \quad and \;\; r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Solution

Centre of mass will be in between the line joining the two masses as shown in figure



let coordinates of centre of mass C be (0,0) thus vector r_1 will be negative and vector r_2 is positive.

Thus $m_1r_1 = m_2 r_2$

$$r_1 = \frac{m_2}{m_1} r_2$$

also $r = r_1 + r_2$

$$\therefore r = \frac{m_2}{m_1} r_2 + r_2$$

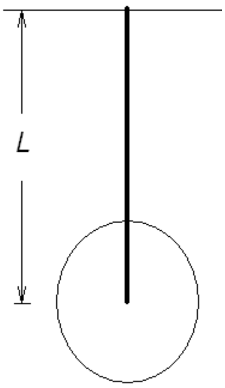
$$\therefore r = \frac{m_2 + m_1}{m_1} r_2$$

$$r_2 = r\left[\frac{m_1}{m_1 + m_2}\right]$$

Similarly it can be obtained

$$r_1 = r\left[\frac{m_2}{m_1 + m_2}\right]$$

Q) A thin rod of length L and uniform cross-section is suspended vertically as shown in figure. A circular disc is attached at the lower end of the road such that the lower end of the rod is at the centre of the disc. Find the position of C.M. of the system with respect to the point of suspension. Let M_1 and M_2 be the masses of the rod and the disc respectively



Solution :

As shown in figure centre of rod must be at the distance L/2 from the point of suspension. And centre of mass of disc is at distance L from the point of suspension

Suppose centre of mass is at distance r_{cm} from the point of suspension then

$$\vec{r}_{CM} = \frac{M_1 \frac{L}{2} + M_2 L}{M_1 + M_2}$$

Difference between Centre of mass (CM) and centre of gravity (CG)

The center of gravity is based on weight, whereas the center of mass is based on mass. So, when the gravitational field across an object is uniform, the two are identical. However, when the object enters a spatially-varying gravitational field, the CG will move closer to regions of the object in a stronger field, whereas the CM is unmoved.

More practically, the CG is the point over which the object can be perfectly balanced; the net torque due to gravity about that point is zero. In contrast, the CM is the average location of the mass distribution. If the object were given some angular momentum, it would spin about the CM.

Clearly if gravitational acceleration is uniform $r_{Cm} = r_{CG}$

If gravitational field is not uniform $r_{Cm} \neq r_{CG}$

Velocity of centre of mass

$$\vec{v}_{CM} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Momentum of centre of mass

$$\vec{P} = M\vec{v}_{CM} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \cdots + m_n\vec{v}_n$$

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

Acceleration of centre of mass

$$\vec{a}_{CM} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

Force on centre of mass

$$\vec{F} = M\vec{a}_{CM} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + \cdots + m_n\vec{a}_n$$

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n$$

Equation shows that the system moves under the influence of the resultant external force F as if the whole mass of the system is concentrated at its centre of mass

Law of conservation of momentum

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots + \vec{P}_n$$

Above equation shows that "If resultant external force acting on a system of particle is zero, then the total linear momentum of the system remain constant" this statement is known as the law of conservation of linear momentum.

Solved numerical

Q) A man weighing 70kg is standing at the centre of a flat boat of mass 350 kg. The man who is at a distance of 10m from the shore walks 2m towards it and stops. How far will he be from the bank? Assume the boat to be of uniform thickness and neglect friction between boat and water.

Solution

Man and boat form a system. This system is not acted by any external force

Thus according to law of conservation of momentum

Centre of mass of system will remain unchanged with reference to observer on the bank

Now man is standing at the centre of boat thus CM is at 10 m from bank

$$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$10 = \frac{70x_1 + 350x_2}{70 + 350}$$

$420= 7x_1 + 35 x_2$

$60= x_1 + 5x_2$ ---- eq(1)

Since man has walked 2 m distance between the CM of Boat and Man is 2m

Also $x_1 – x_2 = 2$ -----eq(2)

From equation (1) and (2) we get

$X_1 = 25/3=8.33$ m

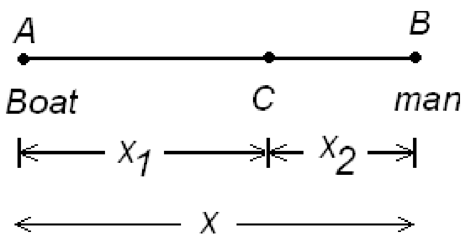
Q) A person is standing on a stationary raft in a lake. The distance of the person from bank of the lake is 30m. The masses of the person and the raft are 60kg and 40kg respectively. Now, the person starts running on the raft towards the bank at the speed of 10m/s with respect to the raft. How far from the bank, would be the person be after one second

Solution

Since no external force acts on the system, the position of centre of mass of the system should remain unchanged. If we take coordinates as (0,0) then

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

Here x_{Cm} = (0,0)



Let m_1 = mass of boat ,

m_2 = mass of man

$$0 = \frac{40(-x_1) + 60x_2}{m_1 + m_2}$$

$$40x_1 = 60x_2$$

Let $x_1 + x_2 = x$

$x_1 = x - x_2$

$$40(x - x_2) = 60x_2$$

$$40x = 100x_2$$

$$x_2 = \frac{2}{5}x$$

So centre of mass C is at a distance of (2/5) from the centre of boat

In one second, person will run through a distance of 10m towards the bank.

So raft must move (2/5)×10 away from the bank. So that centre of mass of system remains unchanged.

So, person distance from bank = 30 - (2/5)×10 = 26 m

Q) Two skaters A and B of mass M and 1.5M are standing together on a frictionless ice surface. They push each other apart. The skater B moves away from A with a speed of 2m/s relative to ice. What will be the separation between two skaters after 8 seconds?

Solution.

Since no external force acts on the system, Centre of mass with respect to observer on ice remains unchanged.

Thus $Mv = (1.5M)\times 2$

Thus velocity of A with respect to ice = 3 m/s

Now relative velocity = 3 +2 m/s

∴ Relative separation = 5×8 = 40 m