**Motion of electron in mutually perpendicular electric field and magnetic field, when electron is released from rest**

Q) An electron released from the origin at a place where a uniform electric field and a uniform magnetic field $B$ exist along the negative $y$-axis and negative $z$-axis respectively. Find the displacement of electron along the $y$-axis when its velocity becomes perpendicular to the electric field for the first.

Solution

Magnetic field $\vec{B} = -B_0 \hat{k}$ and Electric field $\vec{E} = -E_0 \hat{j}$, initial velocity is zero

i) Initial force is along +$y$ axis as electrons moves opposite to direction of electric filed and $\vec{F}_E = e\vec{E}$

ii) When velocity $V > 0$, magnetic force acts and given by $\vec{F}_B = e(\vec{v} \times \vec{B})$ direction of the force is perpendicular to direction of velocity of electron and direction of $B$. Thus electron moves in X-Y plane, Now electric force is always positive Y-axis and magnetic force is always perpendicular to velocity. Such motion is represented in figure below.

![Diagram of electron motion](image)

Speed of electron changes due to Electric filed while direction changes due to Magnetic and electric field both

At point $P$ velocity along $V_y = 0$, thereafter electron moves in negative $y$-axis and comes to rest. At point $P$ velocity is perpendicular to electric field

Then electron again start to move along positive $Y$-axis and repeat its motion

iii) Motion of electron can be given by equation $V = -V_x \hat{i} + V_y \hat{j}$

Now total force $\vec{F}$ due to electric and magnetic filed is

$$\vec{F} = e \vec{E} + e(\vec{v} \times \vec{B})$$

$$\vec{F} = -eE_0 \hat{j} + e\left([-V_x \hat{i} + V_y \hat{j}] \times (-B_0 \hat{k})\right)$$

$$\vec{F} = -eE_0 \hat{j} + e\left([-V_x B_0 \hat{i} - V_y B_0 \hat{j}]\right)$$
\[ \vec{F} = -e(V_yB_0)i - e[E_0 - V_xB_0]j \]

Now Force \( \vec{F} = m\vec{a} \) and \( \vec{a} = a_xi + a_yj \)

\[ m(a_x \dot{i} + a_y \dot{j}) = -e(V_yB_0)i - e[E_0 - V_xB_0] \]

\[ m(a_x) = -e(V_yB_0) \]

\[ m\left(\frac{dV_x}{dt}\right) \dot{i} = -e(V_yB_0)i - - - (i) \]

And

\[ m(a_y) = -e[E_0 - V_xB_0] \]

\[ m\left(\frac{dV_y}{dt}\right) \dot{j} = -e[E_0 - V_xB_0] - - - (ii) \]

Now to solve above equation we will take derivative with respect to time, As \( E_0 \) is constant derivative is zero

\[ m\left(\frac{d^2V_y}{dt^2}\right) \dot{j} = e \left[ \frac{dV_x}{dt}B_0 \right] \dot{j} \]

Substituting value of \( \frac{dV_x}{dt} \) from (i)

\[ m\left(\frac{d^2V_y}{dt^2}\right) \dot{j} = e \left[ -\frac{e}{m}(V_yB_0)B_0 \right] \dot{j} \]

\[ \frac{d^2V_y}{dt^2} = -\left(\frac{e^2B_0^2}{m^2}\right) V_y \]

Above equation is equation of oscillation thus\( \omega^2 = \frac{e^2B_0^2}{m^2} \); \( \therefore \omega = \frac{eB_0}{m} \)

Now Solution to (ii) is

\[ V_y = V_m \sin(\omega t + \delta) \]

When \( t = 0 \), \( V_y = 0 \) Thus \( \delta = 0 \), (called boundary condition)

\[ \therefore V_y = V_m \sin(\omega t) ---(iii) \]

By taking derivative of above equation we get

\[ \frac{dV_y}{dt} = \omega V_m \cos(\omega t) \]

\[ m \frac{dV_y}{dt} = m\omega V_m \cos(\omega t) - - - (iv) \]

From (ii) and (iv) we get

\[ m\omega V_m \cos(\omega t) = -e[E_0 - V_xB_0] \]

\[ eV_xB_0 = eE_0 + m\omega V_m \cos(\omega t) - - - (v) \]
At \( t = 0 \), \( V_x = 0 \) from above equation

\[
e E_0 = -m\omega V_m
\]

\[
V_m = -\frac{eE_0}{m\omega}
\]

As value of \( \omega = \frac{eB_0}{m} \)

Thus

\[
V_m = -\frac{eE_0}{m\omega} = -\frac{eE_0}{m} \times \frac{m}{eB_0} = -\frac{E_0}{B_0}
\]

Substituting value of \( V_m \) in (iii) we get

\[
V_y = \frac{E_0}{B_0} \sin\omega t
\]

\[
\frac{dy}{dt} = \frac{E_0}{B_0} \sin\omega t
\]

\[
dy = -\left(\frac{E_0}{B_0} \sin\omega t\right) dt
\]

Integrating we get

\[
\int_0^t dy = \int_0^t -\left(\frac{E_0}{B_0} \sin\omega t\right) dt
\]

\[
y = -\frac{E_0}{\omega B_0} [\cos\omega t]_0^t
\]

\[
y = -\frac{E_0}{\omega B_0} [\cos\omega t - 1]
\]

\[
y = \frac{E_0}{\omega B_0} [1 - \cos\omega t]
\]

For maximum displacement along \( y \) value of \( \cos\omega t \) should be minimum that is \( \cos\omega t = -1 \)

\[
y_{\text{max}} = 2\frac{E_0}{\omega B_0}
\]

At \( y_{\text{max}} \) velocity of electron is perpendicular to electric field

We can calculate displacement along \( x \) axis as follows

Now from (v) and substituting value of \( \omega \) and \( V_m \) we get

Equation (v) is

\[
e V_x B_0 = eE_0 + m\omega V_m \cos(\omega t)
\]

\[
e V_x B_0 = eE_0 + m \frac{B_0}{m} \left( -\frac{eE_0}{B_0} \right) \cos(\omega t)
\]
\( V_x B_0 = E_0 - E_0 \cos(\omega t) \)

\( V_x = \frac{E_0}{B_0} (1 - \cos \omega t) \)

\( \frac{dx}{dt} = \frac{E_0}{B_0} (1 - \cos \omega t) \)

\( dx = \frac{E_0}{B_0} (1 - \cos \omega t) dt \)

Integrating above equation we get

\( x = \frac{E_0}{B_0} \left[ t - \frac{\sin \omega t}{\omega} \right] \)

We have derived following equations

\( \omega = \frac{eB_0}{m} \)

\( y = \frac{E_0}{\omega B_0} [1 - \cos \omega t] \)

\( x = \frac{E_0}{B_0} \left[ t - \frac{\sin \omega t}{\omega} \right] \)

\( V_y = \frac{E_0}{B_0} \sin \omega t \)

\( V_x = \frac{E_0}{B_0} (1 - \cos \omega t) \)