Motion of electron in mutually perpendicular electric field and magnetic filed, when electron is released from rest

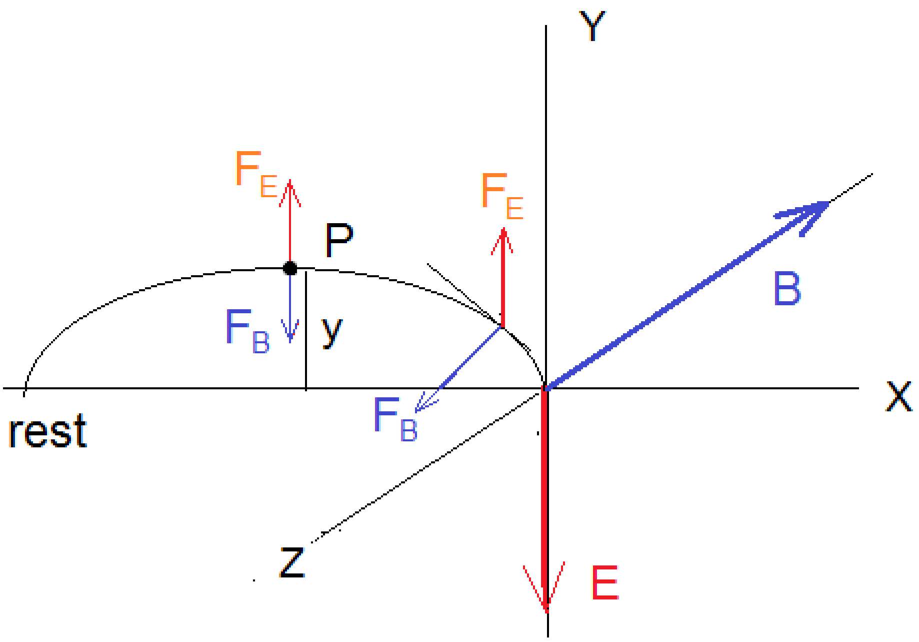
Q) An electron released from the origin at a place where a uniform electric field and a uniform magnetic field B exist along the negative y-axis and negative z-axis respectively. Find the displacement of electron along the y-axis when its velocity becomes perpendicular to the electric filed for the first.

Solution

Magnetic filed $\vec{B} = -B_0\hat{k}$ and Electric filed$\vec{E} = -E_0\hat{j}$, initial velocity is zero

i) Initial force is along +y axis as electrons moves opposite to direction of electric filed and $\overrightarrow{F_E} = e\vec{E}$

ii) When velocity V> 0, magnetic force acts and given by $\overrightarrow{F_B} = e(\vec{v} \times \vec{B})$ direction of the force is perpendicular to direction of velocity of electron and direction of B. Thus electron moves in X-Y plane, Now electric force is always positive Y-axis and magnetic force is always perpendicular to velocity. Such motion is represented in figure below.



Speed of electron changes due to Electric filed while direction changes due to Magnetic and electric field both

At point P velocity along $V_y = 0$, thereafter electron moves in negative y-axis and comes to rest. At point P velocity is perpendicular to electric field

Then electron again start to move along positive Y-axis and repeat its motion

iii) Motion of electron can be given by equation $V = -V_x\hat{\imath} + V_y\hat{\jmath}$

Now total force F due to electric and magnetic filed is

$$\vec{F} = e\vec{E} + e(\vec{v} \times \vec{B})$$

$$\vec{F} = -eE_0\hat{\jmath} + e[(-V_x\hat{\imath} + V_y\hat{\jmath}) \times (-B_0\hat{k})]$$

$$\vec{F} = -eE_0\hat{\jmath} + e[-V_yB_0\hat{\imath} - V_xB_0\hat{\jmath}]$$

$$\vec{F} = -e(V_y B_0)\hat{\imath} - e[E_0 - V_x B_0]\hat{\jmath}$$

Now Force $\vec{F} = m\vec{a}$ and $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$

$$m(a_x\hat{\imath} + a_y\hat{\jmath}) = -e(V_y B_0)\hat{\imath} - e[E_0 - V_x B_0]\hat{\jmath}$$

$$m(a_x\hat{\imath}) = -e(V_y B_0)\hat{\imath}$$

$$m\left(\frac{dV_x}{dt}\right)\hat{\imath} = -e(V_y B_0)\hat{\imath} - - - (i)$$

And

$$m(a_y\hat{\jmath}) = -e[E_0 - V_x B_0]\hat{\jmath}$$

$$m\left(\frac{dV_y}{dt}\right)\hat{\jmath} = -e[E_0 - V_x B_0]\hat{\jmath} - - - (ii)$$

Now to solve above equation we will take derivative with respect to time, As E_0 is constant derivative is zero

$$m\left(\frac{d^2V_y}{dt^2}\right)\hat{\jmath} = e\left[\frac{dV_x}{dt}B_0\right]\hat{\jmath}$$

Substituting value of $\frac{dV_x}{dt}$ from (i)

$$m\left(\frac{d^2V_y}{dt^2}\right)\hat{\jmath} = e\left[-\frac{e}{m}(V_y B_0)B_0\right]\hat{\jmath}$$

$$\left(\frac{d^2V_y}{dt^2}\right) = -\left(\frac{e^2B_0^2}{m^2}\right)V_y$$

Above equation is equation of oscillation thus $\omega^2 = \frac{e^2B_0^2}{m^2}$ $\therefore \omega = \frac{eB_0}{m}$

Now Solution to (ii) is

$V_y = V_m \sin(\omega t + \delta)$

When t=0. $V_y = 0$ Thus $\delta = 0$, (called boundary condition)

$$\therefore V_y = V_m \sin(\omega t) ---(iii)$$

By taking derivative of above equation we get

$$\frac{dV_y}{dt} = \omega V_m cos(\omega t)$$

$$m\frac{dV_y}{dt} = m\omega V_m cos(\omega t) - - - (iv)$$

From (ii) and (iv) we get

$$m\omega V_m cos(\omega t) = -e[E_0 - V_x B_0]$$

$$eV_x B_0 = eE_0 + m\omega V_m cos(\omega t) - - - (v)$$

At t = 0, V_x = 0 from above equation

$$eE_0 = -m\omega V_m$$

$$V_m = -\frac{eE_0}{m\omega}$$

As value of $\omega = \frac{eB_0}{m}$

Thus

$$V_m = -\frac{eE_0}{m\omega} = -\frac{eE_0}{m} \times \frac{m}{eB_0} = -\frac{E_0}{B_0}$$

Substituting value of V_m in (iii) we get

$$V_y = -\frac{E_0}{B_0} sin\omega t$$

$$\frac{dy}{dt} = -\frac{E_0}{B_0} sin\omega t$$

$$dy = -\left(\frac{E_0}{B_0} sin\omega t\right) dt$$

Integrating we get

$$\int_0^y dy = \int_0^t -\left(\frac{E_0}{B_0} sin\omega t\right) dt$$

$$y = -\frac{E_0}{\omega B_0}[cos\omega t]_0^t$$

$$y = -\frac{E_0}{\omega B_0}[cos\omega t - 1]$$

$$y = \frac{E_0}{\omega B_0}[1 - cos\omega t]$$

For maximum displacement along y value of cosωt should be minimum that is cosωt= -1

$$y_{max} = \frac{2E_0}{\omega B_0}$$

At y_{max} velocity of electron is perpendicular to electric filed

We can calculate displacement along x axis as follows

Now from (v) and substituting value of ω and V_m we get

Equation (v) is

$$eV_x B_0 = eE_0 + m\omega V_m cos(\omega t)$$

$$eV_x B_0 = eE_0 + m\frac{eB_0}{m}\left(-\frac{eE_0}{B_0}\right) cos(\omega t)$$

$$V_x B_0 = E_0 - E_0 cos(\omega t)$$

$$V_x = \frac{E_0}{B_0}(1 - cos\omega t)$$

$$\frac{dx}{dt} = \frac{E_0}{B_0}(1 - cos\omega t)$$

$$dx = \frac{E_0}{B_0}(1 - cos\omega t)dt$$

Integrating above equation we get

$$x = \frac{E_0}{B_0}\left[t - \frac{sin\omega t}{\omega}\right]$$

We have derived following equations

$$\omega = \frac{eB_0}{m}$$

$$y = \frac{E_0}{\omega B_0}[1 - cos\omega t]$$

$$x = \frac{E_0}{B_0}\left[t - \frac{sin\omega t}{\omega}\right]$$

$$V_y = -\frac{E_0}{B_0} sin\omega t$$

$$V_x = \frac{E_0}{B_0}(1 - cos\omega t)$$